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Hans Triebel

Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration

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Preface

Although this book deals with some selected topics of the theory of function spaces and the indicated applications, we tried to make it independently readable. For this purpose we provide in Chapter 1 notation and basic facts, give detailed references and prove some specific assertions.

Chapters 2 and 3 deal with Haar bases and Faber bases in function spaces of type B^s_{pq} and F^s_{pq}, covering some (fractional) Sobolev spaces, (classical) Besov spaces and Hölder–Zygmund spaces. In higher dimensions preference is given to several types of spaces with dominating mixed smoothness. This paves the way to study in Chapters 4 and 5 sampling and numerical integration for corresponding spaces on cubes and more general domains. It is well known that numerical integration is symbiotically related to discrepancy, the theory of irregularities of distribution of points, preferably in cubes. This is subject of Chapter 6.

Formulas are numbered within chapters. Furthermore in each chapter all definitions, theorems, propositions, corollaries and remarks are jointly and consecutively numbered. Chapter n is divided in sections $n.k$ and subsections $n.k.l$. But when quoted we refer simply to Section $n.k$ or Section $n.k.l$ instead of Section $n.k$ or Subsection $n.k.l$. If there is no danger of confusion (which is mostly the case) we write A^s_{pq}, $S^r_{pq}A$, …, a^s_{pq}, $s^r_{pq}a$ …(spaces) instead of $A^s_{p,q}$, $S^r_{p,q}A$, …, $a^s_{p,q}$, $s^r_{p,q}a$ …. Similarly a_{jm}, λ_{jm}, Q_{km} (functions, numbers, rectangles) instead of $a_{j,m}$, $\lambda_{j,m}$, $Q_{k,m}$ etc. References ordered by names, not by labels, which roughly coincides, but may occasionally cause minor deviations. The number(s) behind ► in the Bibliography mark the page(s) where the corresponding entry is quoted. log is always taken to base 2. All unimportant positive constants will be denoted by c (with additional marks if there are several c's in the same formula). Our use of $\sim$ (equivalence) is explained on p. 176.

It is a pleasure to acknowledge the great help I have received from my colleagues and friends who made valuable suggestions which have been incorporated in the text. I am especially indebted to Dorothee D. Haroske for her remarks and for producing all the figures and to Erich Novak for many stimulating discussions.

Jena, Spring 2010 Hans Triebel

Contents

Chapter 1
Function spaces

1.1 Isotropic spaces

1.1.1 Definitions

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean n-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. Let $S(\mathbb{R}^n)$ be the usual Schwartz space and $S'(\mathbb{R}^n)$ be the space of all tempered distributions on $\mathbb{R}^n$. Furthermore, $L_p(\mathbb{R}^n)$ with $0 < p \le \infty$, is the standard quasi-Banach space with respect to the Lebesgue measure in $\mathbb{R}^n$, quasi-normed by

$$\|f \,|L_p(\mathbb{R}^n)\| = \Big(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \Big)^{1/p} \tag{1.1}$$

with the natural modification if $p = \infty$. As usual, $\mathbb{Z}$ is the collection of all integers; and $\mathbb{Z}^n$ where $n \in \mathbb{N}$, denotes the lattice of all points $m = (m_1, \dots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. Let $\mathbb{N}_0^n$, where $n \in \mathbb{N}$, be the set of all multi-indices,

$$\alpha = (\alpha_1, \dots, \alpha_n) \quad \text{with } \alpha_j \in \mathbb{N}_0 \text{ and } |\alpha| = \sum_{j=1}^n \alpha_j. \tag{1.2}$$

If $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$ then we put

$$x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} \quad \text{(monomials)}. \tag{1.3}$$

If $\varphi \in S(\mathbb{R}^n)$ then

$$\widehat{\varphi}(\xi) = (F\varphi)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} \varphi(x) \,\mathrm{d}x, \quad \xi \in \mathbb{R}^n, \tag{1.4}$$

denotes the Fourier transform of φ. As usual, $F^{-1}\varphi$ and $\varphi^\vee$ stand for the inverse Fourier transform, given by the right-hand side of (1.4) with i in place of $-i$. Here $x\xi$ denotes the scalar product in $\mathbb{R}^n$. Both F and F^{-1} are extended to $S'(\mathbb{R}^n)$ in the standard way. Let $\varphi_0 \in S(\mathbb{R}^n)$ with

$$\varphi_0(x) = 1 \text{ if } |x| \le 1 \quad \text{and} \quad \varphi_0(y) = 0 \text{ if } |y| \ge 3/2, \tag{1.5}$$

and let

$$\varphi_k(x) = \varphi_0\big(2^{-k}x\big) - \varphi_0\big(2^{-k+1}x\big), \quad x \in \mathbb{R}^n, \; k \in \mathbb{N}. \tag{1.6}$$

Since

$$\sum_{j=0}^\infty \varphi_j(x) = 1 \quad \text{for } x \in \mathbb{R}^n, \tag{1.7}$$

the φ_j form a dyadic resolution of unity. The entire analytic functions $(\varphi_j \hat{f})^\vee(x)$ make sense pointwise in $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$.

Definition 1.1. Let $\varphi = \{\varphi_j\}_{j=0}^\infty$ be the above dyadic resolution of unity.

(i) Let

$$0 < p \le \infty, \quad 0 < q \le \infty, \ s \in \mathbb{R}. \tag{1.8}$$

Then $B^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \,|B^s_{pq}(\mathbb{R}^n)\|_\varphi = \Big(\sum_{j=0}^\infty 2^{jsq} \|(\varphi_j \hat{f})^\vee |L_p(\mathbb{R}^n)\|^q \Big)^{1/q} < \infty \tag{1.9}$$

(with the usual modification if $q = \infty$).

(ii) Let

$$0 < p < \infty, \quad 0 < q \le \infty, \ s \in \mathbb{R}. \tag{1.10}$$

Then $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \,|F^s_{pq}(\mathbb{R}^n)\|_\varphi = \Big\| \Big(\sum_{j=0}^\infty 2^{jsq} \big|(\varphi_j \hat{f})^\vee(\cdot)\big|^q \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\| < \infty \tag{1.11}$$

(with the usual modification if $q = \infty$).

Remark 1.2. The theory of these spaces and their history may be found in [T83], [T92], [T06]. In particular these spaces are independent of admitted resolutions of unity φ according to (1.5)–(1.7) (equivalent quasi-norms). This justifies our omission of the subscript φ in (1.9), (1.11) in the sequel. We remind the reader of a few special cases and properties referring for details to the above books, especially Section 1.2 in [T06].

(i) Let $1 < p < \infty$. Then

$$L_p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n) \tag{1.12}$$

is a well-known *Littlewood–Paley theorem.*

(ii) Let $1 < p < \infty$ and $k \in \mathbb{N}_0$. Then

$$W^k_p(\mathbb{R}^n) = F^k_{p,2}(\mathbb{R}^n) \tag{1.13}$$

are the *classical Sobolev spaces* usually equivalently normed by

$$\|f \,|W^k_p(\mathbb{R}^n)\| = \Big(\sum_{|\alpha| \le k} \|D^\alpha f \,|L_p(\mathbb{R}^n)\|^p \Big)^{1/p}. \tag{1.14}$$

This generalises (1.12).

(iii) Let $\sigma \in \mathbb{R}$. Then

$$I_\sigma : \ f \mapsto \big(\langle \xi \rangle^\sigma \hat{f} \big)^\vee \quad \text{with } \langle \xi \rangle = (1 + |\xi|^2)^{1/2} \tag{1.15}$$

is a one-to-one map of $S(\mathbb{R}^n)$ onto itself and of $S'(\mathbb{R}^n)$ onto itself. Furthermore, I_σ is a lift for the spaces $A^s_{pq}(\mathbb{R}^n)$ with $A = B$ or $A = F$ and $s \in \mathbb{R}$, $0 < p \le \infty$ ($p < \infty$ for the F-scale), $0 < q \le \infty$,

$$I_\sigma A^s_{pq}(\mathbb{R}^n) = A^{s-\sigma}_{pq}(\mathbb{R}^n) \tag{1.16}$$

(equivalent quasi-norms). With

$$H^s_p(\mathbb{R}^n) = I_{-s} L_p(\mathbb{R}^n), \quad s \in \mathbb{R},\ 1 < p < \infty, \tag{1.17}$$

one has

$$H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n), \qquad s \in \mathbb{R},\ 1 < p < \infty, \tag{1.18}$$

and

$$H^k_p(\mathbb{R}^n) = W^k_p(\mathbb{R}^n), \qquad k \in \mathbb{N}_0,\ 1 < p < \infty. \tag{1.19}$$

Nowadays one calls $H^s_p(\mathbb{R}^n)$ *Sobolev spaces* (sometimes fractional Sobolev spaces or Bessel-potential spaces) with the classical Sobolev spaces $W^k_p(\mathbb{R}^n)$ as special cases.

(iv) We denote

$$\mathcal{C}^s(\mathbb{R}^n) = B^s_{\infty\infty}(\mathbb{R}^n), \quad s \in \mathbb{R}, \tag{1.20}$$

as *Hölder–Zygmund spaces*. Let

$$(\Delta^1_h f)(x) = f(x+h) - f(x), \quad (\Delta^{l+1}_h f)(x) = \Delta^1_h(\Delta^l_h f)(x), \tag{1.21}$$

where $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$, $l \in \mathbb{N}$, be the iterated differences in $\mathbb{R}^n$. Let $0 < s < m \in \mathbb{N}$. Then

$$\| f \,|\mathcal{C}^s(\mathbb{R}^n)\|_m = \sup_{x \in \mathbb{R}^n} |f(x)| + \sup |h|^{-s} |\Delta^m_h f(x)| \tag{1.22}$$

where the second supremum is taken over all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$ with $0 < |h| \le 1$, are equivalent norms in $\mathcal{C}^s(\mathbb{R}^n)$.

(v) This assertion can be generalised as follows. Once more let $0 < s < m \in \mathbb{N}$ and $1 \le p, q \le \infty$. Then

$$\| f \,|B^s_{pq}(\mathbb{R}^n)\|_m = \| f \,|L_p(\mathbb{R}^n)\| + \Big(\int_{|h| \le 1} |h|^{-sq} \big\| \Delta^m_h f \,|L_p(\mathbb{R}^n)\big\|^q \frac{\mathrm{d}h}{|h|^n} \Big)^{1/q} \tag{1.23}$$

and

$$\| f \,|B^s_{pq}(\mathbb{R}^n)\|^*_m = \| f \,|L_p(\mathbb{R}^n)\| + \Big(\int_0^1 t^{-sq} \sup_{|h| \le t} \big\| \Delta^m_h f \,|L_p(\mathbb{R}^n)\big\|^q \frac{dt}{t} \Big)^{1/q} \tag{1.24}$$

(with the usual modification if $q = \infty$) are equivalent norms in $B^s_{pq}(\mathbb{R}^n)$. These are the *classical Besov spaces*. We refer to [T92, Chapter 1] and [T06, Chapter 1], where one finds the history of these spaces, further special cases and classical assertions. In addition, (1.23), (1.24) remain to be equivalent quasi-norms in

$$B^s_{pq}(\mathbb{R}^n) \quad \text{with } 0 < p, q \le \infty \ \text{ and } \ s > n\big(\max\big(\tfrac{1}{p}, 1\big) - 1\big), \tag{1.25}$$

[T83, Theorem 2.5.12, p. 110].

1.1.2 Atoms

We give a detailed description of atomic representations of the spaces introduced in Definition 1.1 adapted to our later needs.

Let Q_{jm} be cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-j}m$ with side-length 2^{-j} where $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$. If Q is a cube in $\mathbb{R}^n$ and $r > 0$ then rQ is the cube in $\mathbb{R}^n$ concentric with Q and with side-length r times of the side-length of Q. Let χ_{jm} be the characteristic function of Q_{jm} and

$$\chi_{jm}^{(p)}(x) = 2^{jn/p}\chi_{jm}(x), \quad x \in \mathbb{R}^n,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{1.26}$$

its p-normalised modification where $0 < p \le \infty$.

Definition 1.3. Let $0 < p \le \infty$, $0 < q \le \infty$. Then b_{pq} is the collection of all sequences

$$\lambda = \left\{\lambda_{jm} \in \mathbb{C} : j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\right\} \tag{1.27}$$

such that

$$\|\lambda \,|b_{pq}\| = \Big(\sum_{j=0}^{\infty}\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_{jm}|^p\Big)^{q/p}\Big)^{1/q} < \infty \tag{1.28}$$

and f_{pq} is the collection of all sequences λ according to (1.27) such that

$$\|\lambda \,| f_{pq}\| = \Big\|\Big(\sum_{j,m}|\lambda_{jm}\chi_{jm}^{(p)}(\cdot)|^q\Big)^{1/q}|L_p(\mathbb{R}^n)\Big\| < \infty \tag{1.29}$$

with the usual modification if $p = \infty$ and/or $q = \infty$.

Remark 1.4. One has $b_{pp} = f_{pp}$, $0 < p \le \infty$.

Definition 1.5. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$ and $d > 1$. Then the L_∞-functions $a_{jm}\colon \mathbb{R}^n \mapsto \mathbb{C}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ are called (s,p)-*atoms* if

$$\operatorname{supp} a_{jm} \subset d\, Q_{jm}, \quad j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n; \tag{1.30}$$

there exist all (classical) derivatives $D^\alpha a_{jm}$ with $|\alpha| \le K-1$ such that

$$|D^\alpha a_{jm}(x)| \le 2^{-j(s-\frac{n}{p})+j|\alpha|}, \quad |\alpha| \le K-1,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{1.31}$$

$$|D^\alpha a_{jm}(x) - D^\alpha a_{jm}(y)| \le 2^{-j(s-\frac{n}{p})+jK}|x-y|, \quad |\alpha| = K-1,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{1.32}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, and

$$\int_{\mathbb{R}^n} x^\beta a_{jm}(x)\,\mathrm{d}x = 0, \quad |\beta| < L,\ j \in \mathbb{N},\ m \in \mathbb{Z}^n. \tag{1.33}$$

Remark 1.6. No cancellation (1.33) for $a_{0,m}$ is required. Furthermore, if $L = 0$ then (1.33) is empty (no condition). If $K = 0$ then (1.31), (1.32) means that $a_{jm} \in L_\infty(\mathbb{R}^n)$ and $|a_{jm}(x)| \le 2^{-j(s-\frac{n}{p})}$. Of course, the conditions for the above atoms depend not only on s and p, but also on the given numbers K, L, d. But this will only be indicated when extra clarity is required and denoted as $(s,p)_{K,L,d}$-atoms. Otherwise we speak about (s,p)-atoms. If one replaces (1.31), (1.32) by

$$|D^\alpha a_{jm}(x)| \le 2^{-j(s-\frac{n}{p})+j|\alpha|}, \quad |\alpha| \le K, \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \tag{1.34}$$

then the above definition coincides essentially with [T08, Definition 1.5]. But in connection with spline wavelets it is convenient to replace (1.34) with $|\alpha| = K$ by (1.32) assuming that the derivatives of order $K-1$ are only subject to the indicated Lipschitz conditions. This is immaterial for the following atomic representation theorem. Let as usual

$$\sigma_p = n\Big(\max\Big(\frac{1}{p},1\Big)-1\Big), \quad \sigma_{pq} = n\Big(\max\Big(\frac{1}{p},\frac{1}{q},1\Big)-1\Big). \tag{1.35}$$

Theorem 1.7. (i) *Let $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. Let $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$, $d \in \mathbb{R}$ with*

$$K > s, \quad L > \sigma_p - s, \tag{1.36}$$

and $d > 1$ be fixed. Then $f \in S'(\mathbb{R}^n)$ belongs to $B^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} \lambda_{jm}\, a_{jm} \tag{1.37}$$

where a_{jm} are (s,p)-atoms (more precisely $(s,p)_{K,L,d}$-atoms) according to Definition 1.5 and $\lambda \in b_{pq}$. Furthermore,

$$\| f \,|B^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|b_{pq}\| \tag{1.38}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (1.37) (for fixed K, L, d).

(ii) *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. Let $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$, $d \in \mathbb{R}$ with*

$$K > s, \quad L > \sigma_{pq} - s, \tag{1.39}$$

and $d > 1$ be fixed. Then $f \in S'(\mathbb{R}^n)$ belongs to $F^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented by (1.37) where a_{jm} are (s,p)-atoms (more precisely $(s,p)_{K,L,d}$-atoms) according to Definition 1.5 and $\lambda \in f_{pq}$. Furthermore,

$$\| f \,|F^s_{pq}(\mathbb{R}^n)\| \sim \inf \|\lambda \,|f_{pq}\| \tag{1.40}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (1.37) (for fixed K, L, d).

Remark 1.8. Recall that dQ_{jm} are cubes centred at $2^{-j}m$ with side-length $d\,2^{-j}$ where $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$. For fixed d with $d > 1$ and $j \in \mathbb{N}_0$ there is some overlap of the cubes dQ_{jm} where $m \in \mathbb{Z}^n$. This makes clear that Theorem 1.7 based on Definition 1.5 is reasonable. Otherwise the above formulation with (1.34) in place of (1.31), (1.32) coincides essentially with [T06, Section 1.5.1]. There one finds also technical comments how the convergence in (1.37) must be understood. The replacement of (1.34) by (1.31), (1.32) is justified by the more general assertion in [T06, Corollary 1.23] which in turn is based on [TrW96], [ET96], but essentially also covered by [FrJ85], [FrJ90]. Otherwise we refer to [T92, Section 1.9] where we described the rather involved history of atoms in function spaces.

1.1.3 Local means

Assertions for local means are dual to atomic representations according to Theorem 1.7 as far as smoothness assumptions and cancellation properties are concerned. We rely on [T08, Section 1.13] where we developed a corresponding theory. Let Q_{jm} be the same cubes in $\mathbb{R}^n$ as in the previous Section 1.1.2.

Definition 1.9. Let $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ and $C > 0$. Then the L_∞-functions $k_{jm}\colon \mathbb{R}^n \mapsto \mathbb{C}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are called kernels (of local means) if

$$\operatorname{supp} k_{jm} \subset C\, Q_{jm}, \quad j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n; \tag{1.41}$$

there exist all (classical) derivatives $D^\alpha k_{jm}$ with $|\alpha| \le A - 1$ such that

$$|D^\alpha k_{jm}(x)| \le 2^{jn + j|\alpha|}, \quad |\alpha| \le A-1,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{1.42}$$

$$|D^\alpha k_{jm}(x) - D^\alpha k_{jm}(y)| \le 2^{jn+jA}|x-y|, \quad |\alpha| = A-1,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{1.43}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, and

$$\int_{\mathbb{R}^n} x^\beta\, k_{jm}(x)\, \mathrm{d}x = 0, \quad |\beta| < B,\ j \in \mathbb{N},\ m \in \mathbb{Z}^n. \tag{1.44}$$

Remark 1.10. No cancellation (1.44) for $k_{0,m}$ is required. Furthermore, if $B = 0$ then (1.44) is empty (no condition). If $A = 0$ then (1.42), (1.43) means $k_{jm} \in L_\infty(\mathbb{R}^n)$ and $|k_{jm}(x)| \le 2^{jn}$. Compared with Definition 1.5 for atoms we have different normalisations. We adapt the sequence spaces introduced in Definition 1.3 correspondingly. Recall that χ_{jm} are the characteristic functions of the cubes Q_{jm}.

Definition 1.11. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Then $\bar{b}^s_{pq}$ is the collection of all sequences λ according to (1.27) such that

$$\|\lambda\,|\bar{b}^s_{pq}\| = \Big(\sum_{j=0}^\infty 2^{j(s-\frac{n}{p})q} \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^p \Big)^{q/p} \Big)^{1/q} < \infty \tag{1.45}$$

and $\bar{f}^s_{pq}$ is the collection of all sequences λ according to (1.27) such that

$$\|\lambda \,|\bar{f}^s_{pq}\| = \Big\| \Big(\sum_{j,m} 2^{jsq} \, |\lambda_{jm} \, \chi_{jm}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\| < \infty \tag{1.46}$$

with the usual modifications if $p = \infty$ and/or $q = \infty$.

Remark 1.12. The notation $b^s_{pq}$and f^s_{pq} (without bar) will be reserved for a slight modification of the above sequence spaces in connection with wavelet representations. Similarly as in Remark 1.4 one has $\bar{b}^s_{pp} = \bar{f}^s_{pp}$, $0 < p \le \infty$.

Definition 1.13. Let $f \in B^s_{pq}(\mathbb{R}^n)$ where $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Let k_{jm} be the kernels according to Definition 1.9 with $A > \sigma_p - s$ where σ_p is given by (1.35) and $B = 0$. Then

$$k_{jm}(f) = (f, k_{jm}) = \int_{\mathbb{R}^n} k_{jm}(y)\, f(y)\, \mathrm{d}y, \quad j \in \mathbb{N}_0, \; m \in \mathbb{Z}^n, \tag{1.47}$$

are *local means*, considered as dual pairing within $\big(S(\mathbb{R}^n), S'(\mathbb{R}^n)\big)$. Furthermore,

$$k(f) = \big\{ k_{jm}(f) \,:\, j \in \mathbb{N}_0, \; m \in \mathbb{Z}^n \big\}. \tag{1.48}$$

Remark 1.14. Let $\frac{1}{p} + \frac{1}{p'} = 1$ if $1 \le p \le \infty$ and $p' = \infty$ if $0 < p < 1$. Then

$$B^{-s+\sigma_p}_{p'p'}(\mathbb{R}^n) = B^s_{pp}(\mathbb{R}^n)', \quad s \in \mathbb{R}, \; 0 < p < \infty, \tag{1.49}$$

is the dual space of $B^s_{pp}(\mathbb{R}^n)$, [T83, Theorems 2.11.2, 2.11.3]. We refer also to Theorem 1.20 below. Then $A > \sigma_p - s$ justifies (1.47) as a dual pairing. We refer for details to [T08, Remark 1.14].

Theorem 1.15. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Let* k_{jm} *be the kernels according to Definition* 1.9 *where* $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ *with*

$$A > \sigma_p - s, \quad B > s, \tag{1.50}$$

and $C > 0$ *are fixed. Let* $k(f)$ *be as in* (1.47), (1.48). *Then for some* $c > 0$ *and all* $f \in B^s_{pq}(\mathbb{R}^n)$,

$$\|k(f)\,|\bar{b}^s_{pq}\| \le c\, \|f\,|B^s_{pq}(\mathbb{R}^n)\|. \tag{1.51}$$

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Let* k_{jm} *and* $k(f)$ *be the above kernels where* $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ *with*

$$A > \sigma_{pq} - s, \quad B > s, \tag{1.52}$$

and $C > 0$ *are fixed. Then for some* $c > 0$ *and all* $f \in F^s_{pq}(\mathbb{R}^n)$,

$$\|k(f)\,|\bar{f}^s_{pq}\| \le c\, \|f\,|F^s_{pq}(\mathbb{R}^n)\|. \tag{1.53}$$

Remark 1.16. A proof of this theorem with

$$|D^\alpha k_{jm}(x)| \le 2^{jn+j|\alpha|}, \quad |\alpha| \le A,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{1.54}$$

instead of (1.42), (1.43) may be found in [T08, pp. 7–12] based on [Tri08]. This is the same type of replacement as in (1.34) compared with (1.31), (1.32). On this basis one can follow the proof of [T08, Theorem 1.15] without any changes. Then one obtains the above theorem.

1.1.4 Wavelets

In what follows we rely in addition to atoms and local means on wavelets. We collect what we need later on. We suppose that the reader is familiar with wavelets in $\mathbb{R}^n$ of Daubechies type and the related multiresolution analysis. The standard references are [Dau92], [Mal99], [Mey92], [Woj97]. A short summary of what is needed may also be found in [T06, Section 1.7]. We give first a brief description of some basic notation. As usual, $C^u(\mathbb{R})$ with $u \in \mathbb{N}$ collects all complex-valued continuous functions on $\mathbb{R}$ having continuous bounded derivatives up to order u inclusively. Let

$$\psi_F \in C^u(\mathbb{R}), \quad \psi_M \in C^u(\mathbb{R}),\ u \in \mathbb{N}, \tag{1.55}$$

be *real* compactly supported Daubechies wavelets with

$$\int_{\mathbb{R}} \psi_M(x)\, x^v \,\mathrm{d}x = 0 \quad \text{for all } v \in \mathbb{N}_0 \text{ with } v < u. \tag{1.56}$$

Recall that ψ_F is called the *scaling function* (father wavelet) and ψ_M the associated *wavelet* (mother wavelet). We extend these wavelets from $\mathbb{R}$ to $\mathbb{R}^n$ by the usual multiresolution procedure. Let

$$G = (G_1, \ldots, G_n) \in G^0 = \{F, M\}^n, \tag{1.57}$$

which means that G_r is either F or M. Let

$$G = (G_1, \ldots, G_n) \in G^j = \{F, M\}^{n*}, \quad j \in \mathbb{N}, \tag{1.58}$$

which means that G_r is either F or M where $*$ indicates that at least one of the components of G must be an M. Hence G^0 has 2^n elements, whereas G^j with $j \in \mathbb{N}$ has $2^n - 1$ elements. Let

$$\Psi^j_{G,m}(x) = 2^{jn/2} \prod_{r=1}^{n} \psi_{G_r}\big(2^j x_r - m_r\big), \quad G \in G^j,\ m \in \mathbb{Z}^n, \tag{1.59}$$

where (now) $j \in \mathbb{N}_0$. We always assume that ψ_F and ψ_M in (1.55) have L_2-norm 1. *Then*

$$\big\{\Psi^j_{G,m} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n\big\} \tag{1.60}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$ (for any $u \in \mathbb{N}$) and

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda_m^{j,G} \, 2^{-jn/2} \, \Psi_{G,m}^j \tag{1.61}$$

with

$$\lambda_m^{j,G} = \lambda_m^{j,G}(f) = 2^{jn/2} \int_{\mathbb{R}^n} f(x) \, \Psi_{G,m}^j(x) \, dx = 2^{jn/2} \big(f, \Psi_{G,m}^j \big) \tag{1.62}$$

is the corresponding expansion, adapted to our needs, where $2^{-jn/2}\Psi_{G,m}^j$ are uniformly bounded functions (with respect to j and m). In [T08], based on [HaT05], [Tri04], [T06], we dealt in detail with an extension of the L_2-theory to spaces of type B and F, with and without weights, on $\mathbb{R}^n$, the n-torus $\mathbb{T}^n$, smooth and rough domains and manifolds. In what follows we need only corresponding assertions for $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. We give a brief description. First we adapt the sequence spaces introduced in Definition 1.11 to the extra summation over G in (1.60). The characteristic function χ_{jm} of Q_{jm} has the same meaning as there.

Definition 1.17. Let $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$. Then b^s_{pq} is the collection of all sequences

$$\lambda = \big\{ \lambda_m^{j,G} \in \mathbb{C} : j \in \mathbb{N}_0, \; G \in G^j, \; m \in \mathbb{Z}^n \big\} \tag{1.63}$$

such that

$$\|\lambda \,|b^s_{pq}\| = \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q} \sum_{G \in G^j} \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_m^{j,G}|^p \Big)^{q/p} \Big)^{1/q} < \infty \tag{1.64}$$

and f^s_{pq} is the collection of all sequences λ in (1.63) such that

$$\|\lambda \,| f^s_{pq}\| = \Big\| \Big(\sum_{j,G,m} 2^{jsq} |\lambda_m^{j,G} \, \chi_{jm}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\| < \infty \tag{1.65}$$

with the usual modification if $p = \infty$ and/or $q = \infty$.

The wavelets $\Psi_{G,m}^j$ in (1.59), (1.61) may serve as atoms according to Definition 1.5 (appropriately normalised) and $\lambda_m^{j,G}(f)$ in (1.62) as local means as introduced in Definitions 1.9, 1.13. Then one can ask under which circumstances the Theorems 1.7, 1.15 can be applied. Otherwise we use standard notation naturally extended from Banach spaces to quasi-Banach spaces. In particular, $\{b_j\}_{j=1}^{\infty} \subset B$ in a separable complex quasi-Banach space B is called a *basis* if any $b \in B$ can be uniquely represented as

$$b = \sum_{j=1}^{\infty} \lambda_j \, b_j, \quad \lambda_j \in \mathbb{C} \quad \text{(convergence in } B\text{)}. \tag{1.66}$$

A basis $\{b_j\}_{j=1}^{\infty}$ is called an *unconditional basis* if for any rearrangement σ of $\mathbb{N}$ (one-to-one map of $\mathbb{N}$ onto itself) $\{b_{\sigma(j)}\}_{j=1}^{\infty}$ is again a basis and

$$b = \sum_{j=1}^{\infty} \lambda_{\sigma(j)} \, b_{\sigma(j)} \quad \text{(convergence in } B) \tag{1.67}$$

for any $b \in B$ with (1.66). Standard bases of separable sequence spaces as considered in this book are always unconditional. A basis in a separable quasi-Banach space which is not unconditional is called a *conditional basis*. We refer to [AlK06] for details about bases in Banach (sequence) spaces. As justified at the beginning of [T06, Section 3.1.3] we abbreviate the right-hand side of (1.61) in what follows by

$$\sum_{j,G,m} \lambda_m^{j,G} \, 2^{-jn/2} \, \Psi_{G,m}^j \tag{1.68}$$

since the conditions for the sequences λ always ensure that the corresponding series converges unconditionally at least in $S'(\mathbb{R}^n)$, which means that any rearrangement converges in $S'(\mathbb{R}^n)$ and has the same limit. *Local convergence* in $B^{\sigma}_{pq}(\mathbb{R}^n)$ means convergence in $B^{\sigma}_{pq}(K)$ for any ball K in $\mathbb{R}^n$. Similarly for $F^{\sigma}_{pq}(\mathbb{R}^n)$. Recall that σ_p and σ_{pq} are given by (1.35).

Theorem 1.18. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, *and let* $\Psi^j_{G,m}$ *be the wavelets* (1.59) *based on* (1.55), (1.56) *with*

$$u > \max(s, \sigma_p - s). \tag{1.69}$$

Let $f \in S'(\mathbb{R}^n)$. *Then* $f \in B^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j,G,m} \lambda_m^{j,G} \, 2^{-jn/2} \, \Psi_{G,m}^j, \quad \lambda \in b^s_{pq}, \tag{1.70}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ *and locally in any space* $B^{\sigma}_{pq}(\mathbb{R}^n)$ *with* $\sigma < s$. *The representation* (1.70) *is unique,*

$$\lambda_m^{j,G} = 2^{jn/2} \big(f, \Psi_{G,m}^j \big) \tag{1.71}$$

and

$$I : f \mapsto \big\{ 2^{jn/2} \big(f, \Psi_{G,m}^j \big) \big\} \tag{1.72}$$

is an isomorphic map of $B^s_{pq}(\mathbb{R}^n)$ *onto* b^s_{pq}. *If, in addition,* $p < \infty$, $q < \infty$, *then* $\big\{ \Psi^j_{G,m} \big\}$ *is an unconditional basis in* $B^s_{pq}(\mathbb{R}^n)$.

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, *and*

$$u > \max(s, \sigma_{pq} - s). \tag{1.73}$$

Let $f \in S'(\mathbb{R}^n)$. Then $f \in F^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j,G,m} \lambda^{j,G}_m \, 2^{-jn/2} \, \Psi^j_{G,m}, \quad \lambda \in f^s_{pq}, \tag{1.74}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ and locally in any space $F^\sigma_{pq}(\mathbb{R}^n)$ with $\sigma < s$. The representation (1.74) is unique with (1.71). Furthermore, I in (1.72) is an isomorphic map of $F^s_{pq}(\mathbb{R}^n)$ onto f^s_{pq}. If, in addition, $q < \infty$ then $\{\Psi^j_{G,m}\}$ is an unconditional basis in $F^s_{pq}(\mathbb{R}^n)$.

Remark 1.19. This theorem with more restrictive assumptions for u in (1.69) and (1.73) may be found in [T06, Section 3.1.3, Theorem 3.5, pp. 153–156], based on [Tri04]. The above version with the optimal conditions for u coincides with [T08, Theorem 1.20, pp. 15–17] and goes back to [Tri08].

1.1.5 Duality and interpolation

We presented so far in detail our main ingredients for what follows, atoms, local means and wavelets. But occasionally we need a few other properties of the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ introduced in Definition 1.1. We give a brief description of two such topics, duality and interpolation, restricting us to what is needed later on.

Duality for the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ is considered in the context of the dual pairing $\big(S(\mathbb{R}^n), S'(\mathbb{R}^n)\big)$. This requires that $S(\mathbb{R}^n)$ is dense in the respective space. We refer to [T83, Section 2.11] for the general background. Let, as usual,

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1 \quad \text{if } 1 \le p \le \infty, \ 1 \le q \le \infty, \tag{1.75}$$

$p' = \infty$ if $p < 1$, and $q' = \infty$ if $q < 1$. Let σ_p be as in (1.35).

Theorem 1.20. (i) *Let $0 < p < \infty$, $0 < q < \infty$, $s \in \mathbb{R}$. Then*

$$B^s_{pq}(\mathbb{R}^n)' = B^{-s+\sigma_p}_{p'q'}(\mathbb{R}^n). \tag{1.76}$$

(ii) *Let $1 < p < \infty$, $1 \le q < \infty$, $s \in \mathbb{R}$. Then*

$$F^s_{pq}(\mathbb{R}^n)' = F^{-s}_{p'q'}(\mathbb{R}^n). \tag{1.77}$$

Remark 1.21. We refer to [T83, Theorem 2.11.2, 2.11.3, pp. 178, 180], complemented by [RuS96, Proposition, p. 20] (as far as $q = 1$ in (1.77) is concerned). There one finds further assertions about dual spaces. For what follows the above theorem is sufficient.

We assume that the reader is familiar with real and complex interpolation. Let $\{A_0, A_1\}$ be an interpolation couple of complex quasi-Banach spaces. Then

$$\big(A_0, A_1\big)_{\theta,q}, \quad 0 < \theta < 1, \ 0 < q \le \infty, \tag{1.78}$$

denotes the real interpolation, whereas $[A_0, A_1]_\theta$ with $0 < \theta < 1$ stands for complex interpolation. The general background may be found in [T78] and [T83, Section 2.4]. The classical complex method is restricted to Banach spaces, going back to the 1960s. Only recently the complex method has been extended to a distinguished class of quasi-Banach spaces in a satisfactory way (including the so-called interpolation property). We refer to the survey [KMM07] with [MeM00] as a forerunner. Fortunately enough this method applies to the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. We restrict ourselves to a few assertions needed later on.

Theorem 1.22. (i) *Let* $p, q, q_0, q_1 \in (0, \infty]$, $0 < \theta < 1$ *and*

$$-\infty < s_0 < s_1 < \infty, \quad s = (1-\theta)s_0 + \theta s_1. \tag{1.79}$$

Then

$$\big(B^{s_0}_{pq_0}(\mathbb{R}^n), B^{s_1}_{pq_1}(\mathbb{R}^n)\big)_{\theta,q} = B^s_{pq}(\mathbb{R}^n). \tag{1.80}$$

(ii) *Let* $p_0, p_1, q_0, q_1 \in (0, \infty)$, $s_0 \in \mathbb{R}$, $s_1 \in \mathbb{R}$, $0 < \theta < 1$ *and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad s = (1-\theta)s_0 + \theta s_1. \tag{1.81}$$

Then

$$\big[B^{s_0}_{p_0q_0}(\mathbb{R}^n), B^{s_1}_{p_1q_1}(\mathbb{R}^n)\big]_\theta = B^s_{pq}(\mathbb{R}^n) \tag{1.82}$$

and

$$\big[F^{s_0}_{p_0q_0}(\mathbb{R}^n), F^{s_1}_{p_1q_1}(\mathbb{R}^n)\big]_\theta = F^s_{pq}(\mathbb{R}^n). \tag{1.83}$$

Remark 1.23. If $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$ then

$$B^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow F^s_{pq}(\mathbb{R}^n) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n). \tag{1.84}$$

Hence, if $p < \infty$ then one can replace on the left-hand side of (1.80) one or both B by F. These assertions are cornerstones of the interpolation theory for isotropic function spaces of this type. We refer to [T83, Theorem 2.4.2, p. 64]. The extended complex interpolation formulas (1.82), (1.83) go back to [KMM07, Theorems 5.2, 9.1, pp. 137, 157] and the literature mentioned there, especially [MeM00]. It is of interest that both (1.82) and (1.83) remain valid if $\min(q_0, q_1) < \max(q_0, q_1) = \infty$. If all p's and q's are strictly between 1 and ∞ then the assertions of the theorem are known since a long time. One may consult [T78, Section 2.4] and the literature mentioned there including our own contributions [Tri73a], [Tri73b].

1.1.6 Spaces on domains

In our notation *domain* means open set. Let Ω be a domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Then $L_p(\Omega)$ with $0 < p \le \infty$ is the standard quasi-Banach space of all complex-valued Lebesgue measurable functions f in Ω such that

$$\|f \,|L_p(\Omega)\| = \Big(\int_\Omega |f(x)|^p \,\mathrm{d}x\Big)^{1/p} \tag{1.85}$$

(with the natural modification if $p = \infty$) is finite. As usual, $D(\Omega) = C_0^\infty(\Omega)$ stands for the collection of all complex-valued infinitely differentiable functions in $\mathbb{R}^n$ with compact support in Ω, considered as functions in Ω. Let $D'(\Omega)$ be the dual space of all distributions in Ω. Let $g \in S'(\mathbb{R}^n)$. Then we denote by $g|\Omega$ its *restriction* to Ω,

$$g|\Omega \in D'(\Omega) : \quad (g|\Omega)(\varphi) = g(\varphi) \text{ for } \varphi \in D(\Omega). \tag{1.86}$$

With $A = B$ or $A = F$ the spaces $A^s_{pq}(\mathbb{R}^n)$ have the same meaning as in Definition 1.1.

Definition 1.24. Let Ω be a domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ and let $0 < p \le \infty$ ($p < \infty$ for the F-spaces), $0 < q \le \infty$, $s \in \mathbb{R}$.

(i) Then

$$A^s_{pq}(\Omega) = \big\{ f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in A^s_{pq}(\mathbb{R}^n) \big\}, \tag{1.87}$$

$$\| f \,|A^s_{pq}(\Omega)\| = \inf \|g \,|A^s_{pq}(\mathbb{R}^n)\|, \tag{1.88}$$

where the infimum is taken over all $g \in A^s_{pq}(\mathbb{R}^n)$ with $g|\Omega = f$.

(ii) Let

$$\widetilde{A}^s_{pq}(\bar{\Omega}) = \big\{ f \in A^s_{pq}(\mathbb{R}^n) : \operatorname{supp} f \subset \bar{\Omega} \big\}. \tag{1.89}$$

Then

$$\widetilde{A}^s_{pq}(\Omega) = \big\{ f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in \widetilde{A}^s_{pq}(\bar{\Omega}) \big\}, \tag{1.90}$$

$$\| f \,|\widetilde{A}^s_{pq}(\Omega)\| = \inf \|g \,|A^s_{pq}(\mathbb{R}^n)\|, \tag{1.91}$$

where the infimum is taken over all $g \in \widetilde{A}^s_{pq}(\bar{\Omega})$ with $g|\Omega = f$.

(iii) Let, in addition, Ω be bounded. Then $C(\Omega)$ is the collection of all complex-valued continuous functions in $\bar{\Omega}$.

Remark 1.25. Part (i) is the usual definition of the quasi-Banach space $A^s_{pq}(\Omega)$ by restriction. The spaces $\widetilde{A}^s_{pq}(\bar{\Omega})$ are closed subspaces of $A^s_{pq}(\mathbb{R}^n)$. One has a one-to-one correspondence between $\widetilde{A}^s_{pq}(\bar{\Omega})$ and $\widetilde{A}^s_{pq}(\Omega)$, written in a logically somewhat sloppy way as

$$\widetilde{A}^s_{pq}(\bar{\Omega}) = \widetilde{A}^s_{pq}(\Omega), \tag{1.92}$$

if, and only if,

$$\big\{ h \in A^s_{pq}(\mathbb{R}^n) : \operatorname{supp} h \subset \partial\Omega \big\} = \{0\}. \tag{1.93}$$

This is the case if Ω is a bounded Lipschitz domain (as defined below), $0 < p, q \le \infty$ and $s > \sigma_p$, (1.35), since $|\partial\Omega| = 0$ and $A^s_{pq}(\mathbb{R}^n) \subset L_1^{\mathrm{loc}}(\mathbb{R}^n)$. In (iii) we assumed that Ω is bounded. Then $C(\Omega)$ is a Banach space, furnished as usual with the norm

$$\| f \,|C(\Omega)\| = \sup_{x \in \Omega} |f(x)| = \max_{x \in \bar{\Omega}} |f(x)|. \tag{1.94}$$

For $2 \le n \in \mathbb{N}$,

$$\mathbb{R}^{n-1} \ni x' \mapsto h(x') \in \mathbb{R} \tag{1.95}$$

is called a Lipschitz function (on $\mathbb{R}^{n-1}$) if there is a number $c > 0$ such that

$$|h(x') - h(y')| \le c\,|x' - y'| \quad \text{for all } x' \in \mathbb{R}^{n-1},\ y' \in \mathbb{R}^{n-1}. \tag{1.96}$$

Definition 1.26. Let $2 \le n \in \mathbb{N}$.

(i) A special Lipschitz domain (or graph Lipschitz domain) in $\mathbb{R}^n$ is the collection of all $x = (x', x_n) \in \mathbb{R}^n$ with $x' \in \mathbb{R}^{n-1}$ such that $h(x') < x_n < \infty$ where $h(x')$ is a Lipschitz function according to (1.95), (1.96).

(ii) A bounded Lipschitz domain in $\mathbb{R}^n$ is a bounded domain Ω in $\mathbb{R}^n$ where the boundary $\Gamma = \partial\Omega$ can be covered by finitely many open balls B_j in $\mathbb{R}^n$ with $j = 1, \dots, J$, centred at Γ such that

$$B_j \cap \Omega = B_j \cap \Omega_j \quad \text{for } j = 1, \dots, J, \tag{1.97}$$

where Ω_j are rotations of suitable special Lipschitz domains in $\mathbb{R}^n$.

The space $A^s_{pq}(\Omega)$ in (1.87) is considered as a subset of $D'(\Omega)$ and re,

$$\mathrm{re}\, g = g|\Omega\colon\ A^s_{pq}(\mathbb{R}^n) \hookrightarrow A^s_{pq}(\Omega), \tag{1.98}$$

according to (1.86) is the linear and bounded *restriction operator*. One of the most fundamental problems in the theory of function spaces is the question of whether there is a linear and bounded *extension operator* ext,

$$\mathrm{ext}\colon\ A^s_{pq}(\Omega) \hookrightarrow A^s_{pq}(\mathbb{R}^n), \quad \mathrm{ext}\, f|\Omega = f. \tag{1.99}$$

By (1.98) this can also be written as

$$\mathrm{re} \circ \mathrm{ext} = \mathrm{id} \quad (\text{identity in } A^s_{pq}(\Omega)\ ). \tag{1.100}$$

We say that ext is a *common extension operator* for all

$$B^s_{pq}(\Omega) \quad \text{with } (s, p, q) \in R_B \subset \mathbb{R} \times (0, \infty] \times (0, \infty] \tag{1.101}$$

if

$$\mathrm{dom}\,(\mathrm{ext}) = \bigcup_{(s,p,q)\in R_B} B^s_{pq}(\Omega) \tag{1.102}$$

such that each restriction to an admitted space $B^s_{pq}(\Omega)$ satisfies (1.99) with $A = B$. Similarly for F-spaces or both B-spaces and F-spaces. An extension operator is called *universal* if it is a common extension operator for all spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$.

Theorem 1.27. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$, according to Definition* 1.26 (ii) *or a bounded interval in $\mathbb{R}$. Then there is a universal extension operator for all spaces $A^s_{pq}(\Omega)$ introduced in Definition* 1.24 (i).

Remark 1.28. This assertion is due to V. S. Rychkov, [Ryc99b]. Otherwise the construction of extension operators is one of the most distinguished problems in the theory of function spaces and may be found in [T78], [T83], [T92], [T06]. As for the well-known standard construction for the classical Sobolev spaces $W^k_p(\Omega)$, $1 < p < \infty$, $k \in \mathbb{N}$, we refer to [HaT08]. In [T08, Chapter 4] we dealt with extension problems for spaces of the above type in rough domains (beyond bounded Lipschitz domains). But this is not needed in what follows.

Closely related to the extension problem is the search for intrinsic characterisations of the spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ according to Definition 1.24. In case of bounded Lipschitz domains one has rather satisfactory answers. We refer in particular to [T06, Theorems 1.118, 1.122, pp. 74, 77] and the extensive literature mentioned there. In [T08, Chapter 4] we dealt not only with intrinsic descriptions by classical means (derivatives and differences) but also in terms of atoms and wavelets. We restrict ourselves here to a few assertions which will be of some use later on.

Let $1 < p < \infty$ and $k \in \mathbb{N}_0$. Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Then

$$W^k_p(\Omega) = \{ f \in D'(\Omega) : \| f \,| W^k_p(\Omega)\| < \infty \}, \tag{1.103}$$

$$\| f \,| W^k_p(\Omega)\| = \Big(\sum_{|\alpha| \le k} \| D^\alpha f \,| L_p(\Omega)\|^p \Big)^{1/p} \tag{1.104}$$

are the classical Sobolev spaces in Ω. In general, $W^k_p(\Omega)$ is not the restriction of $W^k_p(\mathbb{R}^n)$ from (1.13), (1.14). But if Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$, according to Definition 1.26 (ii) or a bounded interval in $\mathbb{R}$, then

$$W^k_p(\Omega) = \operatorname{re} W^k_p(\mathbb{R}^n), \quad 1 < p < \infty, \ k \in \mathbb{N}_0, \tag{1.105}$$

is the restriction of $W^k_p(\mathbb{R}^n) = F^k_{p,2}(\mathbb{R}^n)$ to Ω. This can be rephrased as an intrinsic description of $F^k_{p,2}(\Omega)$ according to Definition 1.24 (i). The standard book about Sobolev spaces in rough domains is [Maz85]. We dealt in [T08, Section 2.5.3, pp. 65–68] with intrinsic wavelet characterisations for $W^k_p(\Omega)$ in arbitrary domains. The assertion (1.105) for bounded Lipschitz domains is very classical and goes back to the 1960s. One may consult [T06, Theorem 1.122, p. 77] and the references given there.

Another case of interest for us is the counterpart of (1.24) for the spaces $B^s_{pq}(\Omega)$ where again Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$, or a bounded interval in $\mathbb{R}$. Let $\Delta^l_h f$ be the differences as introduced in (1.21), where $l \in \mathbb{N}$ and $h \in \mathbb{R}^n$. Let for $x \in \Omega$,

$$(\Delta^l_{h,\Omega} f)(x) = \begin{cases} (\Delta^l_h f)(x) & \text{if } x + kh \in \Omega \text{ for } k = 0, \dots, l, \\ 0 & \text{otherwise}, \end{cases} \tag{1.106}$$

be the differences adapted to Ω. With σ_p as in (1.35) let

$$0 < p, q \le \infty, \quad \sigma_p < s < l \in \mathbb{N}. \tag{1.107}$$

Let $\bar{p} = \max(1, p)$. Then $B^s_{pq}(\Omega)$ is the collection of all $f \in L_{\bar{p}}(\Omega)$ such that

$$\| f \,| L_{\bar{p}}(\Omega)\| + \Big(\int_0^1 t^{-sq} \sup_{|h| \le t} \| \Delta^l_{h,\Omega} f \,| L_p(\Omega)\|^q \frac{dt}{t} \Big)^{1/q} < \infty \tag{1.108}$$

(modification if $q = \infty$) in the sense of equivalent quasi-norms. Since $\bar{p} \ge 1$ one has $L_{\bar{p}}(\Omega) \subset D'(\Omega)$. But otherwise one can replace $L_{\bar{p}}(\Omega)$ by $L_p(\Omega)$ in (1.108) in the sense of equivalent quasi-norms. We refer to [T06, Section 1.11.9] and the literature mentioned there, in particular [Dis03], [DeS93].

Proposition 1.29. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, according to Definition 1.26* (ii) *or a bounded interval in $\mathbb{R}$. Then the real and complex interpolation formulas* (1.80), (1.82), (1.83) *with Ω in place of $\mathbb{R}^n$ remain valid with the same parameters as there.*

Remark 1.30. This follows from the Theorems 1.22, 1.27 and the interpolation property. It has also been mentioned explicitly in [KMM07, Theorem 9.4, p. 159]. One may also consult [T06, Section 1.11.8] and the underlying paper [Tri02].

1.1.7 Limiting embeddings

In connection with sampling and numerical integration we are mainly interested in spaces $A^s_{pq}(\Omega)$ which are continuously embedded in $C(\Omega)$ as introduced in Definition 1.24. Here Ω is a bounded domain in $\mathbb{R}^n$. This suggests to have a closer look at related limiting situations only involving spaces of smoothness zero, $s = 0$, integrability $p = 1$ or $p = \infty$ and their interrelations. We follow [SiT95] restricting some assertions proved there in the context of $\mathbb{R}^n$ to bounded domains Ω in $\mathbb{R}^n$.

(i) Smoothness $s = 0$, integrability $p = 1$ and $u, v \in (0, \infty]$. *Then*

$$B^0_{1,u}(\Omega) \hookrightarrow L_1(\Omega) \hookrightarrow B^0_{1,v}(\Omega) \tag{1.109}$$

if, and only if, $0 < u \leq 1$, $v = \infty$. *Furthermore,*

$$F^0_{1,u}(\Omega) \hookrightarrow L_1(\Omega) \tag{1.110}$$

if, and only if, $0 < u \leq 2$. There is no counterpart of the right-hand side of (1.109) for F-spaces since $L_1(\Omega)$ is not embedded in any space $F^0_{1,v}(\Omega)$ with $0 < v \leq \infty$.

(ii) Smoothness $s = 0$, integrability $p = \infty$ and $u, v \in (0, \infty]$. *Then*

$$B^0_{\infty,u}(\Omega) \hookrightarrow C(\Omega) \hookrightarrow B^0_{\infty,v}(\Omega) \tag{1.111}$$

if, and only if, $0 < u \leq 1$, $v = \infty$. This assertion remains valid if one replaces $C(\Omega)$ in (1.111) by $L_\infty(\Omega)$.

(iii) Smoothness $s = 0$, integrabilities $p = \infty$, $p = 1$ and $u \in (0, \infty]$. *Then*

$$B^0_{\infty,u}(\Omega) \hookrightarrow L_1(\Omega) \tag{1.112}$$

if, and only if, $0 < u \leq 2$.

The if-parts are known since a long time and covered by [T83, Sections 2.5.7, 2.5.8]. The more complicated only-if-parts may be found in [SiT95]. In connection with numerical integration of continuous functions it might be of interest to clarify in which spaces $B^0_{1,q}(\Omega)$ and $F^0_{1,q}(\Omega)$ the space $C(\Omega)$ is continuously embedded, complementing the trivial assertion $C(\Omega) \hookrightarrow L_1(\Omega)$.

Theorem 1.31. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $0 < q \le \infty$. Then*

$$C(\Omega) \hookrightarrow B^0_{1,q}(\Omega) \quad \text{if, and only if,} \quad 2 \le q \le \infty, \tag{1.113}$$

and

$$C(\Omega) \hookrightarrow F^0_{1,q}(\Omega) \quad \text{if, and only if,} \quad 2 \le q \le \infty. \tag{1.114}$$

Proof. Step 1. We begin with a preparation. Let Ω be a ball and let $C_0(\Omega)$ be the collection of all $f \in C(\Omega)$ as introduced in Definition 1.24 (iii) with $f|\partial\Omega = 0$. Then $D(\Omega)$ is dense in $C_0(\Omega)$. According to [Mall95, Theorem 6.6, p. 97] one has

$$C_0'(\Omega) = M(\Omega) = \{\text{finite complex Radon measures on } \Omega\} \tag{1.115}$$

for the dual of $C_0(\Omega)$. Let $\widetilde{B}^0_{1,q}(\overline{\Omega})$ be as in (1.89) and put (in slight abuse of notation)

$$C_0(\Omega) = C(\overline{\Omega}) = \big\{ f \in C(\mathbb{R}^n) : \operatorname{supp} f \subset \overline{\Omega} \big\}. \tag{1.116}$$

If $1 \le q < \infty$ then $D(\Omega)$ is also dense in $\widetilde{B}^0_{1,q}(\overline{\Omega})$ (localising f by multiplication with cut-off functions, using the continuity of the translation $f(\cdot) \to f(\cdot + h)$, $h \in \mathbb{R}^n$, and mollification). If one assumes that

$$C(\overline{\Omega}) \hookrightarrow \widetilde{B}^0_{1,q}(\overline{\Omega}), \quad 1 \le q < \infty, \tag{1.117}$$

considered as subspaces of $C(\mathbb{R}^n)$ and $B^0_{1,q}(\mathbb{R}^n)$, then it follows by duality that

$$\widetilde{B}^0_{1,q}(\overline{\Omega})' = B^0_{\infty,q'}(\Omega) \hookrightarrow M(\Omega), \quad 1/q + 1/q' = 1. \tag{1.118}$$

Here we used the $\mathbb{R}^n$-duality (1.76) and a standard factor space argument as in [T78, Sections 2.10.5, 4.8.1, pp. 234, 332] (second edition). We prove the converse assuming that the embedding in (1.118) holds. If $f \in D(\Omega)$ then

$$\begin{aligned} \|f \,|\widetilde{B}^0_{1,q}(\overline{\Omega})\| &\sim \sup \big\{ |(f,g)| : \|g\,|B^0_{\infty,q'}(\Omega)\| \le 1 \big\} \\ &\le \sup \big\{ |(f,g)| : \|g\,|M(\Omega)\| \le c \big\} \\ &\sim \|f \,|C(\overline{\Omega})\|. \end{aligned} \tag{1.119}$$

Completion gives (1.117).

Step 2. We prove (1.113). We may assume that Ω is a ball. Let $2 \le q < \infty$. Then (1.118) follows from (1.112) with $u = q'$. This proves (1.117) and by standard arguments the if-part of (1.113). Let $1 \le q < 2$ and let $\mathbb{T} = [-\pi, \pi]$ be the 1-torus. Let $B^0_{\infty,q'}(\mathbb{T})$ be the corresponding periodic Besov space. Recall that any $f \in D'(\mathbb{T})$ can be expanded by its trigonometrical series,

$$f = \sum_{m \in \mathbb{Z}} b_m \, e^{imx}, \quad x \in \mathbb{T}. \tag{1.120}$$

As far as periodic function spaces are concerned we refer to [ST87, Chapter 3] and [T08, Section 1.3]. In particular if this expansion is lacunary,

$$f = \sum_{k=0}^{\infty} a_k \, e^{i2^k x}, \quad x \in \mathbb{T}, \tag{1.121}$$

then

$$f \in B^0_{\infty,q'}(\mathbb{T}) \iff \Big(\sum_{k=0}^{\infty} |a_k|^{q'} \Big)^{1/q'} \sim \| f \,|B^0_{\infty,q'}(\mathbb{T})\| < \infty. \tag{1.122}$$

Let $\{a_k\} \in \ell_{q'}$ but $\{a_k\} \notin \ell_2$. Then f in (1.121) is not a Radon measure. This follows from [Edw82, pp. 235, 251, Section 15.3.1] since $E = \{2^k\}$ is a Sidon set and any measure with (1.121) belongs to $L_2(\mathbb{T})$, hence $\{a_k\} \in \ell_2$. But then there are also compactly supported elements $f \in B^0_{\infty,q'}(\mathbb{R})$ which are not finite Radon measures (hence generating linear and bounded functionals on corresponding spaces of continuous functions). Let

$$g(x) = f(x_1)\,\chi(x'), \quad x = (x_1, x') \in \mathbb{R}^n, \tag{1.123}$$

with the above f and $\chi \in D(\mathbb{R}^{n-1})$, $\chi(x') = 1$ if $|x'| \le 1$. Then g is not a Radon measure in $\mathbb{R}^n$ (since it does not generate a linear and bounded functional on a suitable space of continuous functions). This disproves (1.118), hence (1.117) and also (1.113). Then one obtains the only-if-part of (1.113).

Step 3. We prove (1.114). The if-part follows from (1.12) and

$$C(\Omega) \hookrightarrow L_p(\Omega) = F^0_{p,2}(\Omega) \hookrightarrow F^0_{1,2}(\Omega), \tag{1.124}$$

$1 < p < \infty$, where we used that Ω is bounded and the resulting monotonicity of the B-spaces and F-spaces with respect to p. One may consult [T06, Remark 4.41, p. 227]. The only-if-part is a consequence of (1.113) and

$$F^0_{1,q}(\Omega) \hookrightarrow B^0_{1,q}(\Omega) \quad \text{for } 1 \le q < 2, \tag{1.125}$$

using elementary embeddings, (1.84) or [T83, (3), Section 3.2.4, p. 195]. □

Remark 1.32. By (1.124) the embedding

$$C(\Omega) \hookrightarrow F^0_{1,2}(\Omega) = h_1(\Omega) \tag{1.126}$$

is almost obvious where $h_1(\Omega)$ is the restriction of the inhomogeneous Hardy space $h_1(\mathbb{R}^n)$ to Ω, [T83, Theorem 1, p. 92]. By (1.125) and the above theorem this assertion cannot be improved in the context of the spaces $A^0_{1,q}(\Omega)$.

1.1.8 Spaces of measurable functions

All spaces considered so far are spaces of distributions, subspaces of $S'(\mathbb{R}^n)$ or of $D'(\Omega)$, where Ω is a domain in $\mathbb{R}^n$. This remains to be our major concept in what follows. On the other hand there is an alternative way to deal with function spaces in the context of measurable functions. The spaces $L_p(\mathbb{R}^n)$ with $0 < p < 1$ as introduced at the beginning of Section 1.1.1 cannot be interpreted as distributional spaces within $S'(\mathbb{R}^n)$ or $D'(\mathbb{R}^n)$. On the other hand they may serve as basic spaces for spaces $\mathbf{B}^s_{pq}(\mathbb{R}^n)$ and $\mathbf{F}^s_{pq}(\mathbb{R}^n)$ with $p, q \in (0, \infty]$ ($p < \infty$ for the F-spaces) of positive smoothness $s > 0$. Especially the spaces $\mathbf{B}^s_{pq}(\mathbb{R}^n)$ have some history in approximation theory. But this is not the subject of this book. One may consult [DeL93, Chapter 2], [DeS93], [HeN07] and the literature mentioned there, especially [StO78]. In [T06, Chapter 9] we developed a new approach to these spaces. We return later on (as a by-product of our main considerations) to sampling numbers in the context of these spaces. For this (and only this) reason we describe here the general background.

All spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ according to Definition 1.1 are subspaces of $S'(\mathbb{R}^n)$. Recall that $\{f_j\}_{j=1}^\infty \subset S'(\mathbb{R}^n)$ is said to converge (weakly) to $f \in S'(\mathbb{R}^n)$ if

$$\lim_{j\to\infty} f_j(\varphi) \to f(\varphi) \quad \text{for all } \varphi \in S(\mathbb{R}^n). \tag{1.127}$$

Let μ_L be the Lebesgue measure in $\mathbb{R}^n$ and let $\mathbf{M}(\mathbb{R}^n)$ be the collection of all equivalence classes of all μ_L almost everywhere finite complex-valued functions f in $\mathbb{R}^n$. This is a linear space which can be furnished with the *convergence in measure*. This means that $\{f_j\}_{j=1}^\infty \subset \mathbf{M}(\mathbb{R}^n)$ converges in measure to $f \in \mathbf{M}(\mathbb{R}^n)$ if for any $\varepsilon > 0$,

$$\lim_{j\to\infty} \mu_L\big\{x : |f_j(x) - f(x)| \ge \varepsilon\big\} = 0. \tag{1.128}$$

A detailed discussion of the convergence in measure, also in relation to the convergence almost everywhere, and how to convert $\mathbf{M}(\mathbb{R}^n)$ into a complete metric space may be found in [Mall95, Section I,5]. For our purpose it is sufficient to remark that the convergence in $L_p(\mathbb{R}^n)$, quasi-normed by (1.1), is stronger than the convergence in measure. Let $0 < p < \infty$ and let $\{f_j\}_{j=1}^\infty \subset L_p(\mathbb{R}^n)$ such that $f_j \to f \in L_p(\mathbb{R}^n)$ in $L_p(\mathbb{R}^n)$. Let $\varepsilon > 0$. Then (1.128) follows from

$$\varepsilon\, \mu_L^{1/p}\big\{x : |f_j(x) - f(x)| \ge \varepsilon\big\} \le \Big(\int_{\mathbb{R}^n} |f(x) - f_j(x)|^p \,\mathrm{d}x\Big)^{1/p}. \tag{1.129}$$

One may consider $\mathbf{M}(\mathbb{R}^n)$ as the substitute of $S'(\mathbb{R}^n)$ in the context of the spaces $B^s_{pq}(\mathbb{R}^n)$, $F^s_{pq}(\mathbb{R}^n)$, in what follows. If Ω is a domain in $\mathbb{R}^n$ then one has an obvious counterpart $\mathbf{M}(\Omega)$ of $\mathbf{M}(\mathbb{R}^n)$ which may serve as a substitute of $D'(\Omega)$.

For $f \in \mathbf{M}(\mathbb{R}^n)$, $l \in \mathbb{N}$ and $h \in \mathbb{R}^n$ let as in (1.21)

$$(\Delta^1_h f)(x) = f(x+h) - f(x), \quad (\Delta^{l+1}_h f)(x) = \Delta^1_h(\Delta^l_h f)(x), \quad x \in \mathbb{R}^n, \tag{1.130}$$

be the iterated differences in $\mathbb{R}^n$. Let

$$d^l_{t,p} f(x) = \Big(t^{-n} \int_{|h|\le t} |(\Delta^l_h f)(x)|^p \,\mathrm{d}h\Big)^{1/p}, \quad l \in \mathbb{N},\ 0 < p < \infty, \tag{1.131}$$

$x \in \mathbb{R}^n$, be related ball means. If Ω is a domain (= open set) in $\mathbb{R}^n$ and $g \in \mathbf{M}(\mathbb{R}^n)$ then

$$g|\Omega \in \mathbf{M}\Omega), \quad (g|\Omega)(x) = g(x), \quad x \in \Omega, \tag{1.132}$$

is the restriction of g to Ω.

Definition 1.33. Let $u \in \mathbb{N}$.

(i) Let $0 < p \le \infty$, $0 < q \le \infty$, $0 < s < l \in \mathbb{N}$. Then $\mathbf{B}^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in L_p(\mathbb{R}^n)$ such that

$$\|f \,|\mathbf{B}^s_{pq}(\mathbb{R}^n)\|_l = \|f \,|L_p(\mathbb{R}^n)\| + \left(\int_0^1 t^{-sq} \sup_{|h| \le t} \|\Delta^l_h f \,|L_p(\mathbb{R}^n)\|^q \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty \tag{1.133}$$

with the usual modification if $q = \infty$.

(ii) Let $0 < p < \infty$, $0 < q \le \infty$, $0 < s < l \in \mathbb{N}$. Then $\mathbf{F}^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in L_p(\mathbb{R}^n)$ such that

$$\|f \,|\mathbf{F}^s_{pq}(\mathbb{R}^n)\|_l = \|f \,|L_p(\mathbb{R}^n)\| + \left\| \left(\int_0^1 t^{-sq} d^l_{t,p} f(\cdot)^q \frac{\mathrm{d}t}{t} \right)^{1/q} |L_p(\mathbb{R}^n) \right\| < \infty \tag{1.134}$$

with the usual modification if $q = \infty$.

(iii) Let Ω be a domain (= open set) in $\mathbb{R}^n$. Let $0 < p, q \le \infty$ (with $p < \infty$ for the $\mathbf{F}$-spaces) and $\mathbf{A}^s_{pq}(\mathbb{R}^n)$ be either $\mathbf{B}^s_{pq}(\mathbb{R}^n)$ or $\mathbf{F}^s_{pq}(\mathbb{R}^n)$. Then

$$\mathbf{A}^s_{pq}(\Omega) = \big\{ f \in L_p(\Omega) : f = g|\Omega \text{ for some } g \in \mathbf{A}^s_{pq}(\mathbb{R}^n) \big\} \tag{1.135}$$

and

$$\|f \,|\mathbf{A}^s_{pq}(\Omega)\| = \inf \|g \,|\mathbf{A}^s_{pq}(\mathbb{R}^n)\|, \tag{1.136}$$

where the infimum is taken over all $g \in \mathbf{A}^s_{pq}(\mathbb{R}^n)$ with $g|\Omega = f$.

Remark 1.34. We followed essentially [T06, Section 9.2.2, pp. 386–390] where one finds further explanations. In particular $\mathbf{A}^s_{pq}(\mathbb{R}^n)$, and then also $\mathbf{A}^s_{pq}(\Omega)$, are quasi-Banach spaces. They are independent of $l \in \mathbb{N}$ with $l > s$. This is the counterpart of Definition 1.1. There one asks for which $f \in S'(\mathbb{R}^n)$ one has (1.9), (1.11). Similarly one can reformulate the above definition asking for which $f \in \mathbf{M}(\mathbb{R}^n)$ one has (1.133) or (1.134). If $\mathbf{A}^s_{pq}(\mathbb{R}^n)$ is continuously embedded in some space $L_r(\mathbb{R}^n)$ with $1 \le r \le \infty$, then $\mathbf{A}^s_{pq}(\mathbb{R}^n)$ can also be considered as subspace of $S'(\mathbb{R}^n)$ consisting of regular distributions. Let σ_p be as in (1.35). Then one has in the sense of this interpretation

$$B^s_{pq}(\mathbb{R}^n) = \mathbf{B}^s_{pq}(\mathbb{R}^n), \quad 0 < p, q \le \infty, \; s > \sigma_p. \tag{1.137}$$

As far as limiting cases $s = \sigma_p$ are concerned we refer also to [ScV09]. A corresponding assertion for the F-spaces is a little bit more complicated. We refer to [T06, pp. 388, 389]. In case of the B-spaces one may consult (1.23)–(1.25). Otherwise we collected in [T06, Theorem 1.116, pp. 72/73] what is known about characterisations of $A^s_{pq}(\mathbb{R}^n)$

in terms of differences and ball means of differences. Part (iii) of the above definition is the direct counterpart of Definition 1.24 (i). Then it follows from (1.137) that

$$B^s_{pq}(\Omega) = \mathbf{B}^s_{pq}(\Omega), \quad 0 < p,q \le \infty,\ s > \sigma_p, \tag{1.138}$$

for any domain Ω in $\mathbb{R}^n$. On the other hand for suitable spaces $B^s_{pq}(\Omega)$ and $F^s_{pq}(\Omega)$ in bounded Lipschitz domains there are intrinsic characterisations of these spaces in terms of differences and ball means of differences, [T06, Theorem 1.118, Remark 1.119, pp. 74–76]. In particular with the adapted differences $\Delta^l_{h,\Omega}$ according to (1.106) one has for the spaces $B^s_{pq}(\Omega)$ covered by (1.138) the characterisation (1.108). This can be extended to all spaces $\mathbf{B}^s_{pq}(\Omega)$. Let $(\cdot,\cdot)_{\theta,q}$ be the real interpolation method mentioned in (1.78). For the B-spaces and F-spaces on $\mathbb{R}^n$ we have Theorem 1.22 and for the corresponding spaces on bounded Lipschitz domains Proposition 1.29. It is somewhat doubtful whether one has a full counterpart of these assertions for the **B**-spaces and **F**-spaces on $\mathbb{R}^n$ and on bounded Lipschitz domains. But there is one special case which will be of crucial importance for us later on.

Theorem 1.35. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$, according to Definition* 1.26 (ii) *or a bounded interval in $\mathbb{R}$. Let $\mathbf{B}^s_{pq}(\Omega)$ be the spaces introduced in* (1.135), (1.136) *with $\mathbf{A} = \mathbf{B}$ where $0 < p,q \le \infty$ and $s > 0$.*

(i) *Let $s < l \in \mathbb{N}$ and let $\Delta^l_{h,\Omega} f$ be the differences according to* (1.106). *Then $\mathbf{B}^s_{pq}(\Omega)$ is the collection of all $f \in L_p(\Omega)$ such that*

$$\|f\,|\mathbf{B}^s_{pq}(\Omega)\|_l = \|f\,|L_p(\Omega)\| + \left(\int_0^1 t^{-sq} \sup_{|h|\le t} \left\| \Delta^l_{h,\Omega} f\,|L_p(\Omega)\right\|^q \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty \tag{1.139}$$

(*equivalent quasi-norms*) *with the usual modification if $q = \infty$.*

(ii) *Let $p, q_1, q_2 \in (0,\infty]$. Let $0 < \theta < 1$. Then*

$$\big(L_p(\Omega), \mathbf{B}^s_{pq_1}(\Omega)\big)_{\theta,q_2} = \mathbf{B}^{s\theta}_{pq_2}(\Omega). \tag{1.140}$$

Remark 1.36. If $s > \sigma_p$ then one has (1.138), and (1.139) coincides essentially with (1.108) where one can replace $L_{\bar p}(\Omega)$ by $L_p(\Omega)$. This is reasonable since the above spaces are considered now as subspaces of $L_p(\Omega)$. The full theorem (including in particular the cases with $p < 1$ not covered by (1.138)) follows from [DeS93, Theorem 6.1, p. 858]. In this paper one finds also the remarkable interpolation assertion (1.140), [DeS93, Theorem 6.3, p. 859]. One can replace Ω by $\mathbb{R}^n$, hence

$$\big(L_p(\mathbb{R}^n), \mathbf{B}^s_{pq_1}(\mathbb{R}^n)\big)_{\theta,q_2} = \mathbf{B}^{s\theta}_{pq_2}(\mathbb{R}^n) \tag{1.141}$$

for the same parameters as in part (ii) of the above theorem. We refer in this context also to [T01, pp. 373/374] where we discussed this type of interpolation and where one finds also further related references.

Remark 1.37. Later on we need a few more specific assertions for the spaces $\mathbf{B}^s_{pq}(\mathbb{R}^n)$ (with $n = 1$) and $\mathbf{B}^s_{pq}(\Omega)$. This applies in particular to atomic and quarkonial representations where we rely on [Net89], [HeN07] and [T06, Section 9.2]. More recent results may be found in [HaS08], [Schn09a], [Schn09b], [Schn10]. We mention here only one assertion which will be of some use for us. Recall the well-known embedding

$$B^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow F^s_{pq}(\mathbb{R}^n) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n), \tag{1.142}$$

$s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, [T83, Proposition 2, p. 47]. The counterpart

$$\mathbf{B}^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow \mathbf{F}^s_{pq}(\mathbb{R}^n) \hookrightarrow \mathbf{B}^s_{p,\max(p,q)}(\mathbb{R}^n), \tag{1.143}$$

$s > 0$, $0 < p < \infty$, $0 < q \le \infty$, may be found in [Schn09b, Proposition 1.19(i)].

1.2 Spaces with dominating mixed smoothness

1.2.1 Definitions

A substantial part of this book deals with sampling, numerical integration and discrepancy in the context of spaces with dominating mixed smoothness. In this Section 1.2 we recall basic definitions, collect some assertions and prove a few new properties which will be helpful for what follows. This may complement the huge literature about this topic but it is far from being something like a survey.

Let $L_p(\mathbb{R}^n)$ with $0 < p \le \infty$ be the usual Lebesgue spaces quasi-normed by (1.1) with the natural modification if $p = \infty$. We use the same notation as introduced in Section 1.1.1. *Sobolev spaces with dominating mixed smoothness*

$$S^r_p W(\mathbb{R}^2) = \left\{ f \in L_p(\mathbb{R}^2) : \|f \,|S^r_p W(\mathbb{R}^2)\| < \infty \right\} \tag{1.144}$$

for $1 < p < \infty$, $r \in \mathbb{N}$ and with

$$\begin{aligned} \|f\,|S^r_p W(\mathbb{R}^2)\| = \|f\,|L_p(\mathbb{R}^2)\| + \left\| \frac{\partial^r f}{\partial x_1^r} \,|L_p(\mathbb{R}^2) \right\| + \left\| \frac{\partial^r f}{\partial x_2^r} |L_p(\mathbb{R}^2) \right\| \\ + \left\| \frac{\partial^{2r} f}{\partial x_1^r \partial x_2^r} \,|L_p(\mathbb{R}^2) \right\| \end{aligned} \tag{1.145}$$

go back to K. I. Babenko, [Bab60] (periodic case) and S. M. Nikol'skij, [Nik62], [Nik63]. One may also consult the relevant parts in [Nik77] (first edition 1969) and [BIN75]. It is remarkable that at this time (and even a little bit earlier) N. M. Korobov, [Kor59], and N. S. Bakhvalov, [Bak59], [Bak64] observed that dominating mixed smoothness is an adequate instrument to deal with sampling and numerical integration (one of our main topics later on). A description of these early results may be found in the survey [Tem03, p. 387]. The systematic Fourier-analytical approach to several versions of spaces with dominating mixed smoothness of type $S^r_{pq}B(\mathbb{R}^n)$ and $S^r_{pq}F(\mathbb{R}^n)$ goes back to H.-J. Schmeisser, [Schm80], [Schm82] (his habilitation) and may also be found in [ST87, Chapter 2]. One may consult [ST87, Section 2.1, pp. 80/81] for further historical comments and references covering the substantial early history of these spaces. Finally we refer to [Tem93] and the recent surveys [Schm07], [ScS04], [Vyb06]. Some new characterisations of spaces with dominating mixed smoothness, especially in terms of Peetre's maximal function, may be found in [Baz03], [Baz04], [Baz05a], [Baz05b], [Ull08], [Ull09]. As said we are interested here only in a few more specific aspects. For sake of simplicity we deal mainly with the two-dimensional case. But it will be rather clear how the n-dimensional generalisations look like.

Let $\hat{f}$ and $f^\vee$ be the Fourier transform and its inverse in $S'(\mathbb{R}^2)$ as introduced in Section 1.1.1. Let $\varphi_0 \in S(\mathbb{R})$ with

$$\varphi_0(t) = 1 \text{ if } |t| \le 1 \quad \text{and} \quad \varphi_0(v) = 0 \text{ if } |v| \ge 3/2, \tag{1.146}$$

and let

$$\varphi_l(t) = \varphi_0\big(2^{-l}t\big) - \varphi_0\big(2^{-l+1}t\big), \quad t \in \mathbb{R},\ l \in \mathbb{N}, \tag{1.147}$$

be the one-dimensional resolution of unity according to (1.5)–(1.7). Let

$$\varphi_k(x) = \varphi_{k_1}(x_1)\,\varphi_{k_2}(x_2), \quad k = (k_1, k_2) \in \mathbb{N}_0^2, \quad x = (x_1, x_2) \in \mathbb{R}^2. \tag{1.148}$$

Since

$$\sum_{k \in \mathbb{N}_0^2} \varphi_k(x) = 1 \quad \text{for } x \in \mathbb{R}^2, \tag{1.149}$$

the φ_k form a resolution of unity. Recall that the entire analytic functions $(\varphi_k \hat{f})^\vee(x)$ make sense pointwise in $\mathbb{R}^2$ for any $f \in S'(\mathbb{R}^2)$. The counterpart of Definition 1.1 reads now as follows.

Definition 1.38. Let $\varphi = \{\varphi_k\}_{k \in \mathbb{N}_0^2}$ be the above dyadic resolution of unity.

(i) Let

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad r \in \mathbb{R}. \tag{1.150}$$

Then $S^r_{pq}B(\mathbb{R}^2)$ is the collection of all $f \in S'(\mathbb{R}^2)$ such that

$$\|f \,|S^r_{pq}B(\mathbb{R}^2)\|_\varphi = \Big(\sum_{k \in \mathbb{N}_0^2} 2^{r(k_1+k_2)q} \big\| (\varphi_k \hat{f})^\vee | L_p(\mathbb{R}^2) \big\|^q \Big)^{1/q} < \infty \tag{1.151}$$

(with the usual modification if $q = \infty$).

(ii) Let

$$0 < p < \infty, \quad 0 < q \le \infty, \quad r \in \mathbb{R}. \tag{1.152}$$

Then $S^r_{pq}F(\mathbb{R}^2)$ is the collection of all $f \in S'(\mathbb{R}^2)$ such that

$$\|f \,|S^r_{pq}F(\mathbb{R}^2)\|_\varphi = \Big\| \Big(\sum_{k \in \mathbb{N}_0^2} 2^{r(k_1+k_2)q} |(\varphi_k \hat{f})^\vee(\cdot)|^q \Big)^{1/q} | L_p(\mathbb{R}^2) \Big\| < \infty \tag{1.153}$$

(with the usual modification if $q = \infty$).

Remark 1.39. These are special cases of a larger variety of spaces with dominating mixed smoothness as considered in [ST87, Chapter 2] and the underlying papers, in particular [Schm80], [Schm82]. As for the more or less obvious n-dimensional generalisation one may consult [Vyb06]. These spaces are independent of admitted resolutions of unity φ according to (1.148), (1.149). This justifies our omission of the subscript φ in (1.151), (1.153). We collect a few special cases and properties referring for details to [ST87]. This is the counterpart of Remark 1.2.

(i) Let $1 < p < \infty$. Then

$$L_p(\mathbb{R}^2) = S^0_{p,2}F(\mathbb{R}^2) \tag{1.154}$$

is again a *Littlewood–Paley theorem*.

(ii) Let $1 < p < \infty$ and $r \in \mathbb{N}_0$. Then

$$S^r_p W(\mathbb{R}^2) = S^r_{p,2}F(\mathbb{R}^2) \tag{1.155}$$

are the classical *classical Sobolev spaces with dominating mixed smoothness*, usually equivalently normed by (1.145) and

$$\|f\,|S^r_p W(\mathbb{R}^2)\| \sim \sum_{\substack{\alpha\in\mathbb{N}^2_0 \\ 0\le\alpha_j\le r}} \|D^\alpha f\,|L_p(\mathbb{R}^2)\|. \tag{1.156}$$

This generalises (1.154)

(iii) Let $\sigma \in \mathbb{R}$. Then

$$J_\sigma : f \mapsto \big((1+\xi_1^2)^{\sigma/2}(1+\xi_2^2)^{\sigma/2}\hat{f}\,\big)^\vee \tag{1.157}$$

is a one-to-one map of $S(\mathbb{R}^2)$ onto itself and of $S'(\mathbb{R}^2)$ onto itself. Furthermore, J_σ is a lift for the spaces $S^r_{pq}A(\mathbb{R}^2)$ with $A = B$ or $A = F$ and $r \in \mathbb{R}$, $0 < p \le \infty$ ($p < \infty$ for the F-scale), $0 < q \le \infty$,

$$J_\sigma S^r_{pq}A(\mathbb{R}^2) = S^{r-\sigma}_{pq}A(\mathbb{R}^2), \tag{1.158}$$

(equivalent quasi-norms). With

$$S^r_p H(\mathbb{R}^2) = J_{-r}L_p(\mathbb{R}^2), \quad r \in \mathbb{R},\ 1 < p < \infty, \tag{1.159}$$

one has

$$S^l_p H(\mathbb{R}^2) = S^l_p W(\mathbb{R}^2), \quad l \in \mathbb{N}_0,\ 1 < p < \infty. \tag{1.160}$$

In analogy to (1.18), (1.19) one may call $S^r_p H(\mathbb{R}^2)$ *Sobolev spaces with dominating mixed smoothness*, generalising the above classical Sobolev spaces with dominating mixed smoothness. By (1.154), (1.158), (1.159) one has

$$S^r_p H(\mathbb{R}^2) = S^r_{p,2}F(\mathbb{R}^2), \quad r \in \mathbb{R},\ 1 < p < \infty. \tag{1.161}$$

(iv) We denote

$$S^r\mathcal{C}(\mathbb{R}^2) = S^r_{\infty\infty}B(\mathbb{R}^2), \quad r \in \mathbb{R}, \tag{1.162}$$

as *Hölder–Zygmund spaces with dominating mixed smoothness*. We need the mixed version of the differences as introduced in (1.21), (1.22). Let

$$\begin{aligned} \Delta^1_{h,1} f(x_1,x_2) &= f(x_1+h,x_2) - f(x_1,x_2), \\ \Delta^{l+1}_{h,1} f(x) &= \Delta^1_{h,1}(\Delta^l_{h,1} f)(x), \end{aligned} \tag{1.163}$$

where $x=(x_1,x_2)\in\mathbb{R}^2$, $h\in\mathbb{R}$, $l\in\mathbb{N}$, be the iterated differences with respect to the x_1-direction. Similarly $\Delta^l_{h,2} f$, $l\in\mathbb{N}$. Let

$$\Delta^{l,l}_h f(x) = \Delta^{l,l}_{h_1,h_2} f(x) = \Delta^l_{h_2,2}\big(\Delta^l_{h_1,1} f\big)(x), \quad x\in\mathbb{R}^2, \tag{1.164}$$

be the mixed differences, $l\in\mathbb{N}$, $h=(h_1,h_2)\in\mathbb{R}^2$. Let $0<r<l\in\mathbb{N}$. Then $S^r\mathcal{C}(\mathbb{R}^2)$ is the collection of all complex-valued continuous functions in $\mathbb{R}^2$ such that

$$\begin{aligned} \|f\,|S^r\mathcal{C}(\mathbb{R}^2)\| = {} & \sup_{x\in\mathbb{R}^2} |f(x)| + \sup_{\substack{h_1\in\mathbb{R}\\ x\in\mathbb{R}^2}} |h_1|^{-r}|\Delta^l_{h_1,1} f(x)| \\ & + \sup_{\substack{h_2\in\mathbb{R}\\ x\in\mathbb{R}^2}} |h_2|^{-r}|\Delta^l_{h_2,2} f(x)| + \sup_{\substack{h\in\mathbb{R}^2\\ x\in\mathbb{R}^2}} |h_1h_2|^{-r}|\Delta^{l,l}_h f(x)| \end{aligned} \tag{1.165}$$

is finite (equivalent norms), [ST87, p. 126].

(v) This assertion can be generalised as follows. Let $0<r<l\in\mathbb{N}$ and $1\le p,q\le\infty$. Then $S^r_{pq}B(\mathbb{R}^2)$ is the collection of all $f\in L_p(\mathbb{R}^2)$ such that

$$\begin{aligned} \|f\,|L_p(\mathbb{R}^2)\| &+ \Big(\int_{|h_1|<1} |h_1|^{-rq}\big\|\Delta^l_{h_1,1} f\,|L_p(\mathbb{R}^2)\big\|^q \frac{dh_1}{h_1}\Big)^{1/q} \\ &+ \Big(\int_{|h_2|<1} |h_2|^{-rq}\big\|\Delta^l_{h_2,2} f\,|L_p(\mathbb{R}^2)\big\|^q \frac{dh_2}{h_2}\Big)^{1/q} \\ &+ \Big(\int_{\substack{|h_1|<1\\ |h_2|<1}} |h_1h_2|^{-rq}\big\|\Delta^{l,l}_{h_1,h_2} f\,|L_p(\mathbb{R}^2)\big\|^q \frac{dh_1\,dh_2}{h_1h_2}\Big)^{1/q} \end{aligned} \tag{1.166}$$

is finite (equivalent norms). Similarly, $S^r_{pq}B(\mathbb{R}^2)$ is the collection of all $f\in L_p(\mathbb{R}^2)$ such that

$$\begin{aligned} \|f\,|L_p(\mathbb{R}^2)\| &+ \Big(\int_0^1 t^{-rq} \sup_{|h_1|\le t} \big\|\Delta^l_{h_1,1} f\,|L_p(\mathbb{R}^2)\big\|^q \frac{dt}{t}\Big)^{1/q} \\ &+ \Big(\int_0^1 t^{-rq} \sup_{|h_2|\le t} \big\|\Delta^l_{h_2,2} f\,|L_p(\mathbb{R}^2)\big\|^q \frac{dt}{t}\Big)^{1/q} \\ &+ \Big(\int_0^1\int_0^1 (t_1t_2)^{-rq} \sup_{\substack{|h_1|\le t_1\\ |h_2|\le t_2}} \big\|\Delta^{l,l}_{h_1,h_2} f\,|L_p(\mathbb{R}^2)\big\|^q \frac{dt_1\,dt_2}{t_1t_2}\Big)^{1/q} \end{aligned} \tag{1.167}$$

is finite (equivalent norms). We refer to [ST87, Theorem 2, p. 122]. These are the *classical Besov spaces with dominating mixed smoothness*. There are extensions to some cases with $p < 1$ and/or $q < 1$, [ST87, Theorem 1, p. 118/119], but not a final assertion as in the isotropic case. Further characterisations of $S^r_{pq}B(\mathbb{R}^2)$ and $S^r_{pq}F(\mathbb{R}^2)$ in terms of (ball means) of differences may be found in [Ull06].

1.2.2 Atoms

We describe atomic representations for the spaces $S^r_{pq}A(\mathbb{R}^2)$ according to Definition 1.38 with $A \in \{B, F\}$. This is the counterpart of Section 1.1.2 where we dealt with the atomic representations for the isotropic spaces $A^s_{pq}(\mathbb{R}^n)$.

Let χ_{km} with $k \in \mathbb{N}^2_0$ and $m \in \mathbb{Z}^2$ be the characteristic function of the rectangle

$$Q_{km} = \big(2^{-k_1}m_1,\, 2^{-k_1}(m_1+1)\big) \times \big(2^{-k_2}m_2,\, 2^{-k_2}(m_2+1)\big) \tag{1.168}$$

and let

$$\chi^{(p)}_{km}(x) = 2^{\frac{k_1+k_2}{p}}\chi_{km}(x), \quad k \in \mathbb{N}^2_0,\ m \in \mathbb{Z}^2,\ x \in \mathbb{R}^2, \tag{1.169}$$

be its p-normalised modification where $0 < p \le \infty$. Let dQ_{km} with $d > 0$ be the rectangle concentric with Q_{km} and with side-lengths $d2^{-k_1}$ and $d2^{-k_2}$.

Definition 1.40. Let $0 < p \le \infty$, $0 < q \le \infty$. Then $s_{pq}b$ is the collection of all sequences

$$\lambda = \big\{\lambda_{km} \in \mathbb{C} : k \in \mathbb{N}^2_0,\ m \in \mathbb{Z}^2\big\} \tag{1.170}$$

such that

$$\|\lambda\,|s_{pq}b\| = \Big(\sum_{k\in\mathbb{N}^2_0}\Big(\sum_{m\in\mathbb{Z}^2}|\lambda_{km}|^p\Big)^{q/p}\Big)^{1/q} < \infty, \tag{1.171}$$

and $s_{pq}f$ is the collection of all sequences according to (1.170) such that

$$\|\lambda\,|s_{pq}f\| = \Big\|\Big(\sum_{k,m}|\lambda_{km}\,\chi^{(p)}_{km}(\cdot)|^q\Big)^{1/q}\,|L_p(\mathbb{R}^2)\Big\| < \infty \tag{1.172}$$

with the usual modifications if $p = \infty$ and/or $q = \infty$.

Remark 1.41. This is the counterpart of Definition 1.3. One has $s_{pp}b = s_{pp}f$, $0 < p \le \infty$.

Definition 1.42. Let $r \in \mathbb{R}$, $0 < p \le \infty$, $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$ and $d > 1$. Then the L_∞-functions $a_{km}\colon \mathbb{R}^2 \mapsto \mathbb{C}$ with $k \in \mathbb{N}^2_0$, $m \in \mathbb{Z}^2$ are called *(r,p)-atoms* if

$$\operatorname{supp} a_{km} \subset dQ_{km}, \quad k \in \mathbb{N}^2_0,\ m \in \mathbb{Z}^2; \tag{1.173}$$

there exist all (classical) derivatives $D^\alpha a_{km}$ with $\alpha = (\alpha_1, \alpha_2)$, $0 \le \alpha_j \le K$ such that

$$|D^\alpha a_{km}(x)| \le 2^{-(k_1+k_2)(r-\frac{1}{p})+k_1\alpha_1+k_2\alpha_2}, \quad k \in \mathbb{N}^2_0,\ m \in \mathbb{Z}^2, \tag{1.174}$$

$x \in \mathbb{R}^2$ and

$$\int_{\mathbb{R}} x_j^\beta \, a_{km}(x) \, dx_j = 0, \quad j = 1, 2; \; \beta < L, \; k_j \in \mathbb{N}, \; m \in \mathbb{Z}^2. \tag{1.175}$$

Remark 1.43. This is the counterpart of Definition 1.5. If $L = 0$ then (1.175) is empty (no condition). If $K = 0$ then (1.174) means that $a_{km} \in L_\infty(\mathbb{R}^2)$ and $|a_{km}(x)| \le 2^{-(k_1+k_2)(r-\frac{1}{p})}$. No cancellation (1.175) is required if $k_j = 0$. If $a_{km}(x) = a_{k_1m_1}(x_1)\, a_{k_2m_2}(x_2)$ then one has the same conditions as in Definition 1.5 with $n = 1$. The above definition coincides essentially with [Vyb06, Definition 2.3, p. 25] and [Schm07, Definition 4.4, p. 180] (with different normalisations). One may also consult [Baz05a].

Theorem 1.44. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $r \in \mathbb{R}$. *Let* $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$, $d \in \mathbb{R}$ *with*

$$K > r, \quad L > \max\left(\frac{1}{p}, 1\right) - 1 - r, \tag{1.176}$$

and $d > 1$ *be fixed. Then* $f \in S'(\mathbb{R}^2)$ *belongs to* $S^r_{pq}B(\mathbb{R}^2)$ *if, and only if, it can be represented as*

$$f = \sum_{k \in \mathbb{N}_0^2} \sum_{m \in \mathbb{Z}^2} \lambda_{km} \, a_{km}, \tag{1.177}$$

where a_{km} *are related* (r, p)*-atoms according to Definition* 1.42 *and* $\lambda \in s_{pq}b$. *Furthermore,*

$$\| f \,|S^r_{pq}B(\mathbb{R}^2)\| \sim \inf \|\lambda \,|s_{pq}b\| \tag{1.178}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (1.177) *(for fixed* K, L, d*).*

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $r \in \mathbb{R}$. *Let* $K \in \mathbb{N}_0$, $L \in \mathbb{N}_0$, $d \in \mathbb{R}$ *with*

$$K > r, \quad L > \max\left(\frac{1}{p}, \frac{1}{q}, 1\right) - 1 - r, \tag{1.179}$$

and $d > 1$ *be fixed. Then* $f \in S'(\mathbb{R}^2)$ *belongs to* $S^r_{pq}F(\mathbb{R}^2)$ *if, and only if, it can be represented by* (1.177) *where* a_{km} *are related* (r, p)*-atoms according to Definition* 1.42 *and* $\lambda \in s_{pq}f$. *Furthermore,*

$$\| f \,|S^r_{pq}F(\mathbb{R}^2)\| \sim \inf \|\lambda \,|s_{pq}f\| \tag{1.180}$$

are equivalent quasi-norms where the infimum is taken over all admissible representations (1.177) *(for fixed* K, L, d*).*

Remark 1.45. This is the direct counterpart of Theorem 1.7. It is remarkable that the restrictions for L in (1.176), (1.179) are the same as in (1.36), (1.39) with $n = 1$ in (1.35). This reflects the well known fact that spaces with dominating mixed smoothness behave in many respects, for example embedding theorems, in the same way as the spaces $A^s_{pq}(\mathbb{R})$ on the real line. This will be one of the points later on when we deal with sampling, numerical integration and discrepancy. A detailed proof of the above theorem may be found in [Vyb06, Section 2.2], where one can replace the assumed continuity of a_{km} by $a_{km} \in L_\infty$ without any changes. In Definition 1.5 and, as a consequence, in Theorem 1.7, we weakened a possible differentiability assumption for $D^\alpha a_{jm}$ with $|\alpha| = K$ by the Lipschitz continuity (1.32). This can also be done in Definition 1.42 and Theorem 1.44 if $\alpha_1 = K$ or $\alpha_2 = K$ in (1.174). For this purpose one has to modify the proof in [Vyb06] slightly. We refer in this context also to [Schm07, Theorem 4.6, p. 182]. In both papers, [Vyb06], [Schm07], more general spaces with dominating mixed smoothness are considered. Finally we remark that the above theorem remains valid (at least for some K, L, d) if one admits only atoms

$$a_{km}(x) = a_{k_1m_1}(x_1)\, a_{k_2m_2}(x_2), \quad k = (k_1, k_2) \in \mathbb{N}_0^2,\ m = (m_1, m_2) \in \mathbb{Z}^2, \tag{1.181}$$

having a product structure. This follows from wavelet representations of the above spaces, interpreting wavelets, satisfying (1.181), as optimal atoms.

1.2.3 Local means

We describe the counterpart of Section 1.1.3 where we dealt with local means for isotropic spaces. The kernels we are looking for are products of one-dimensional kernels according to Definition 1.9 which will be now denoted by w_{jm} in order to avoid a clash of notation with the summation index k. Let

$$Q_{km} = I_{k_1m_1} \times I_{k_2m_2}, \quad k = (k_1, k_2) \in \mathbb{N}_0^2,\ m = (m_1, m_2) \in \mathbb{Z}^2. \tag{1.182}$$

be the rectangles according to (1.168).

Definition 1.46. Let $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ and $C > 0$. Let

$$W_{km}(x) = w^1_{k_1m_1}(x_1)\, w^2_{k_2m_2}(x_2), \quad k \in \mathbb{N}_0^2,\ m \in \mathbb{Z}^2,\ x \in \mathbb{R}^2, \tag{1.183}$$

where $w^j_{k_jm_j} : \mathbb{R} \mapsto \mathbb{C}$ are L_∞-functions with

$$\operatorname{supp} w^j_{k_jm_j} \subset C\, I_{k_jm_j}, \quad k_j \in \mathbb{N}_0,\ m_j \in \mathbb{Z}, \tag{1.184}$$

$j = 1, 2$. Then W_{km} are called kernels (of local means) if there exist all (classical) derivatives $\frac{d^l}{dt^l} w^j_{k_jm_j}(t)$ with $l \le A$ such that

$$\left| \frac{d^l}{dt^l} w^j_{k_jm_j}(t) \right| \le 2^{k_j(1+l)}, \quad k_j \in \mathbb{N}_0,\ m_j \in \mathbb{Z}, \tag{1.185}$$

$t \in \mathbb{R}$, and if

$$\int_{\mathbb{R}} t^v\, w_{k_jm_j}(t)\, dt = 0, \quad v \in \mathbb{N}_0,\ v < B,\ k_j \in \mathbb{N},\ m_j \in \mathbb{Z}, \tag{1.186}$$

$j = 1, 2$.

Remark 1.47. In other words, the kernels W_{km} for spaces with dominating mixed smoothness are products of corresponding one-dimensional kernels according to Definition 1.9. One can replace (1.185) with $l = A$ by the counterpart of the Lipschitz condition (1.43). Otherwise one has the same interpretations as in Remark 1.10. Next we need the counterparts of the Definitions 1.11, 1.13. We adapt the spaces $s_{pq}b$ and $s_{pq}f$ introduced in Definition 1.40 to the different normalisation of kernels compared with atoms. Let χ_{km} be the characteristic function of Q_{km} in (1.182).

Definition 1.48. Let $r \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$. Then $s^r_{pq}b$ is the collection of all sequences λ according to (1.170) such that

$$\|\lambda \,|s^r_{pq}b\| = \Big(\sum_{k \in \mathbb{N}^2_0} 2^{(k_1+k_2)(r-\frac{1}{p})q} \Big(\sum_{m \in \mathbb{Z}^2} |\lambda_{km}|^p \Big)^{q/p} \Big)^{1/q} < \infty \tag{1.187}$$

and $s^r_{pq}f$ is the collection of all sequences λ according to (1.170) such that

$$\|\lambda \,|s^r_{pq}f\| = \Big\| \Big(\sum_{k,m} 2^{(k_1+k_2)rq} \, |\lambda_{km}\, \chi_{km}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^2) \Big\| < \infty \tag{1.188}$$

with the usual modification if $p = \infty$ and/or $q = \infty$.

Remark 1.49. The summation over k and m in (1.188) is the same as in (1.187), hence $k \in \mathbb{N}^2_0$, $m \in \mathbb{Z}^2$. Furthermore $s^r_{pp}b = s^r_{pp}f$.

Definition 1.50. Let $f \in S^r_{pq}B(\mathbb{R}^2)$ where $r \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$. Let W_{km} be the kernels according to Definition 1.46 with $A > \max\big(\frac{1}{p}, 1\big) - 1 - r$. Then

$$W_{km}(f) = (f, W_{km}) = \int_{\mathbb{R}^2} W_{km}(y)\, f(y)\, \mathrm{d}y, \quad k \in \mathbb{N}^2_0,\ m \in \mathbb{Z}^2, \tag{1.189}$$

are *local means*, considered as dual pairing within $\big(S(\mathbb{R}^2), S'(\mathbb{R}^2)\big)$. Furthermore,

$$W(f) = \big\{ W_{km}(f) : k \in \mathbb{N}^2_0,\ m \in \mathbb{Z}^2 \big\}. \tag{1.190}$$

Remark 1.51. This is the counterpart of Definition 1.13. As there one has to justify that (1.189) makes sense. This is again a matter of duality. The needed counterpart of (1.49) may be found in [Vyb06, p. 42].

Theorem 1.52. (i) *Let* $0 < p \leq \infty$, $0 < q \leq \infty$, $r \in \mathbb{R}$. *Let* W_{km} *be the kernels according to Definition* 1.46 *where* $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ *with*

$$A > \max\Big(\frac{1}{p}, 1\Big) - 1 - r, \quad B > r, \tag{1.191}$$

and $C > 0$ *are fixed. Let* $W(f)$ *be as in* (1.189), (1.190). *Then for some* $c > 0$ *and all* $f \in S^r_{pq}B(\mathbb{R}^2)$,

$$\|W(f)\,|s^r_{pq}b\| \leq c\, \|f\,|S^r_{pq}B(\mathbb{R}^2)\|. \tag{1.192}$$

(ii) *Let* $0 < p < \infty$, $0 < q \leq \infty$, $r \in \mathbb{R}$. *Let* W_{km} *and* $W(f)$ *be the above kernels where* $A \in \mathbb{N}_0$, $B \in \mathbb{N}_0$ *with*

$$A > \max\left(\frac{1}{p}, \frac{1}{q}, 1\right) - 1 - r, \quad B > r, \tag{1.193}$$

and $C > 0$ *are fixed. Then for some* $c > 0$ *and all* $f \in S^r_{pq}F(\mathbb{R}^2)$,

$$\| W(f) \,|s^r_{pq} f \| \leq c \, \| f \,| S^r_{pq} F(\mathbb{R}^2) \|. \tag{1.194}$$

Proof (Outline). This is the counterpart of Theorem 1.15 where we referred in Remark 1.16 for a proof to [T08, pp. 7–12]. One can adapt this proof to the above situation. For this purpose one expands $f \in S^r_{pq}B(\mathbb{R}^2)$ according to Theorem 1.44 (i) where the atoms a_{km} in (1.177) have the product structure (1.181) (one may think about wavelet representations we are going to describe later on). Together with (1.183) one has

$$\int_{\mathbb{R}^2} W_{km}(y)\, a_{\tilde{k}\tilde{m}}(y)\, \mathrm{d}y = \prod_{j=1}^{2} \int_{\mathbb{R}} w^j_{k_j m_j}(y_j)\, a_{\tilde{k}_j \tilde{m}_j}(y_j)\, \mathrm{d}y_j. \tag{1.195}$$

and one can apply the corresponding estimates in [T08, pp. 7–2] with $n = 1$ (one-dimensional). For this purpose one has to adapt some technicalities as it has been done in [Vyb06, Lemma 1.18]. Following this way one obtains part (i) of the above theorem. For the F-spaces we relied in [T08, p. 10, (1.70)] on the well-known vector-valued maximal inequality of Fefferman–Stein. There is a coordinate-wise generalisation which may be found [ST87, Theorem 1, p. 23] with a reference to [Bag75] and which has been used in [Schm07], [Vyb06] in the above context. After this modification one can again follow the arguments in [T08] appropriately adapted. One obtains part (ii) of the above theorem. □

Remark 1.53. Local means for spaces with dominating mixed smoothness have been considered in [Vyb06, Section 1.3.5] under more restrictive assumptions for the kernels. In Theorem 1.15 we relied on (1.42), (1.43) instead of (1.54). Similarly one can replace (1.185) with $l = A$ by a corresponding Lipschitz condition. Based on a related comment in Remark 1.45 one can modify the above arguments. Then one obtains Theorem 1.52 under these slightly weaker assumptions.

1.2.4 Wavelets

We are interested in the counterpart of Theorem 1.18 for spaces with dominating mixed smoothness preferring now the same normalisation as for atoms in Definition 1.42 and Theorem 1.44. This is better adapted to our later needs when dealing with Haar bases and Faber bases. Let $\psi_F \in C^u(\mathbb{R})$ and $\psi_M \in C^u(\mathbb{R})$ be the same compactly supported real Daubechies wavelets as in (1.55), (1.56). Let

$$\psi_{-1,m}(t) = \sqrt{2}\, \psi_F(t - m) \quad \text{and} \quad \psi_{km}(t) = \psi_M(2^k t - m) \tag{1.196}$$

with $k \in \mathbb{N}_0$ and $m \in \mathbb{Z}$. Let $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$. We always assume that

$$\big\{2^{k/2}\psi_{km}(t) : k \in \mathbb{N}_{-1},\ m \in \mathbb{Z}\big\} \tag{1.197}$$

is an orthonormal basis in $L_2(\mathbb{R})$. Let $\mathbb{N}^2_{-1} = \mathbb{N}_{-1} \times \mathbb{N}_{-1}$ and let

$$\psi_{km}(x) = \psi_{k_1 m_1}(x_1)\,\psi_{k_2 m_2}(x_2), \quad k = (k_1,k_2) \in \mathbb{N}^2_{-1},\ m = (m_1,m_2) \in \mathbb{Z}^2, \tag{1.198}$$

be the usual tensor product in $\mathbb{R}^2$. Then

$$\big\{2^{\frac{1}{2}(k_1+k_2)}\psi_{km} : k \in \mathbb{N}^2_{-1},\ m \in \mathbb{Z}^2\big\} \tag{1.199}$$

is an orthonormal basis in $L_2(\mathbb{R}^2)$ (for any $u \in \mathbb{N}$). Furthermore,

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{Z}^2} \lambda_{km}\, 2^{\frac{1}{2}(k_1+k_2)}\, \psi_{km} \tag{1.200}$$

is an atomic representation according to (1.177) with $a_{km} = 2^{\frac{1}{2}(k_1+k_2)}\psi_{km}$ and

$$\lambda_{km} = \lambda_{km}(f) = 2^{\frac{1}{2}(k_1+k_2)} \int_{\mathbb{R}^2} f(x)\,\psi_{km}(x)\,\mathrm{d}x. \tag{1.201}$$

We discussed in Section 1.1.4 what is meant by an unconditional basis in a separable quasi-Banach space, unconditional convergence in $S'(\mathbb{R}^2)$ and local convergence now with respect to $S^{\varrho}_{pq}B(\mathbb{R}^2)$. Similarly as in (1.68) we abbreviate (1.200) by

$$\sum_{k,m} \lambda_{km}\, 2^{\frac{1}{2}(k_1+k_2)}\, \psi_{km} \tag{1.202}$$

since the conditions for the sequence λ always ensure that the corresponding series converges unconditionally at least in $S'(\mathbb{R}^2)$. We need the sequence spaces introduced in Definition 1.40 in connection with atoms but with $k \in \mathbb{N}^2_{-1}$ instead of $k \in \mathbb{N}^2_0$. To avoid any misunderstanding we give a precise formulation. Let $0 < p \le \infty$, $0 < q \le \infty$ and

$$\lambda = \big\{\lambda_{km} \in \mathbb{C} : k \in \mathbb{N}^2_{-1},\ m \in \mathbb{Z}^2\big\}. \tag{1.203}$$

Let $\chi^{(p)}_{km}$ be the p-normalised characteristic function (1.169) of the rectangle (1.168) now with respect to $k \in \mathbb{N}^2_{-1}$ and $m \in \mathbb{Z}^2$. Then $s_{pq}b^-$ is the collection of all sequences λ in (1.203) such that

$$\|\lambda\,|s_{pq}b^-\| = \Big(\sum_{k \in \mathbb{N}^2_{-1}} \Big(\sum_{m \in \mathbb{Z}^2} |\lambda_{km}|^p \Big)^{q/p} \Big)^{1/q} < \infty \tag{1.204}$$

and $s_{pq}f^-$ is the collection of all sequences λ in (1.203) such that

$$\|\lambda\,|s_{pq}f^-\| = \Big\| \Big(\sum_{k,m} |\lambda_{km}\,\chi^{(p)}_{km}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^2) \Big\| < \infty \tag{1.205}$$

with the usual modifications if $p = \infty$ and/or $q = \infty$.

Theorem 1.54. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, $r \in \mathbb{R}$, *and let* ψ_{km} *be the wavelets* (1.198) *based on* (1.55), (1.56) *with*

$$u > \max\left(r, \max\left(\frac{1}{p}, 1\right) - 1 - r\right). \tag{1.206}$$

Let $f \in S'(\mathbb{R}^2)$. *Then* $f \in S^r_{pq}B(\mathbb{R}^2)$ *if, and only if, it can be represented as*

$$f = \sum_{k,m} \lambda_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})}\, \psi_{km}, \quad \lambda \in s_{pq}b^-, \tag{1.207}$$

unconditional convergence being in $S'(\mathbb{R}^2)$ *and locally in any space* $S^{\varrho}_{pq}B(\mathbb{R}^2)$ *with* $\varrho < r$. *The representation* (1.207) *is unique,*

$$\lambda_{km} = 2^{(k_1+k_2)(r-\frac{1}{p}+1)}(f, \psi_{km}) \tag{1.208}$$

and

$$J: f \mapsto \left\{2^{(k_1+k_2)(r-\frac{1}{p}+1)}(f, \psi_{km})\right\} \tag{1.209}$$

is an isomorphic map of $S^r_{pq}B(\mathbb{R}^2)$ *onto* $s_{pq}b^-$. *If, in addition,* $p < \infty$, $q < \infty$, *then* $\{\psi_{km}\}$ *is an unconditional basis in* $S^r_{pq}B(\mathbb{R}^2)$.

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$, $r \in \mathbb{R}$, *and*

$$u > \max\left(r, \max\left(\frac{1}{p}, \frac{1}{q}, 1\right) - 1 - r\right). \tag{1.210}$$

Let $f \in S'(\mathbb{R}^2)$. *Then* $f \in S^r_{pq}F(\mathbb{R}^2)$ *if, and only if, it can be represented as*

$$f = \sum_{k,m} \lambda_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})}\, \psi_{km}, \quad \lambda \in s_{pq}f^-, \tag{1.211}$$

unconditional convergence being in $S'(\mathbb{R}^2)$ *and locally in any space* $S^{\varrho}_{pq}F(\mathbb{R}^2)$ *with* $\varrho < r$. *The representation* (1.211) *is unique with* (1.208). *Furthermore,* J *in* (1.209) *is an isomorphic map of* $S^r_{pq}F(\mathbb{R}^2)$ *onto* $s_{pq}f^-$. *If, in addition,* $q < \infty$ *then* $\{\psi_{km}\}$ *is an unconditional basis in* $S^r_{pq}F(\mathbb{R}^2)$.

Proof. This is the counterpart of the corresponding Theorem 1.18 for the isotropic spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ now with the same normalisation as in Theorem 1.44 for atoms. The necessary references may be found in Remark 1.19. The proof of Theorem 1.18 is reduced to the atomic representations in Theorem 1.7 and the local means in Theorem 1.15. For spaces with dominating mixed smoothness the Theorems 1.44, 1.52 are the direct counterparts of these two crucial ingredients. Then one can follow the arguments in the references given in Remark 1.19. One point should be mentioned. Wavelet representations of the above type for sufficiently large $u \in \mathbb{N}$ in (1.55), (1.56) have been proved in [Vyb06, Theorem 2.11, p. 41]. These smooth wavelets can be identified with optimal atoms in (1.181), (1.195) resulting in the sharp Theorem 1.52 as far as the assumptions for A and B are concerned. Afterwards one can use this theorem combined with Theorem 1.44 to minimise the needed smoothness for $u \in \mathbb{N}$ according to (1.206), (1.210). □

Remark 1.55. In [Vyb06, Theorem 2.11, p. 41] and [Schm07, Theorem 4.9] one finds wavelet representations for more general spaces with dominating mixed smoothness and also further related references. Our arguments give optimal restrictions for $u \in \mathbb{N}$ in (1.206), (1.210) which are the same as in the one-dimensional case $n = 1$ in (1.69), (1.73).

1.2.5 Higher dimensions

Not only the definitions of the spaces with dominating mixed smoothness in $\mathbb{R}^2$, but also the special cases and equivalent norms in Section 1.2.1 have obvious counterparts in $\mathbb{R}^n$, $n > 2$. This applies also to the assertions about atoms, local means and wavelets in the preceding sections. We stick at $n = 2$ also in what follows as the model case. But occasionally it is desirable to switch to n-dimensional formulations, for example in connection with discrepancy and logarithmic spaces. For this reason it is helpful to fix a few definitions and to hint how some assertions look like in $\mathbb{R}^n$ where now $n \geq 2$.

(i) *Basic definitions*. With

$$\varphi_k(x) = \prod_{j=1}^{n} \varphi_{k_j}(x_j), \quad k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n, \quad x \in \mathbb{R}^n, \tag{1.212}$$

in place of (1.148) the corresponding spaces $S^r_{pq}B(\mathbb{R}^n)$ and $S^r_{pq}F(\mathbb{R}^n)$ in Definition 1.38 are quasi-normed by

$$\|f \,|S^r_{pq}B(\mathbb{R}^n)\|_\varphi = \Big(\sum_{k \in \mathbb{N}_0^n} 2^{r(k_1+\cdots+k_n)q} \|(\varphi_k \hat{f})^\vee | L_p(\mathbb{R}^n)\|^q \Big)^{1/q} \tag{1.213}$$

and

$$\|f \,|S^r_{pq}F(\mathbb{R}^n)\|_\varphi = \Big\| \Big(\sum_{k \in \mathbb{N}_0^n} 2^{r(k_1+\cdots+k_n)q} |(\varphi_k \hat{f})^\vee(\cdot)|^q \Big)^{1/q} | L_p(\mathbb{R}^n) \Big\|. \tag{1.214}$$

(ii) *Classical Sobolev spaces*. Let $1 < p < \infty$. Then the Littlewood–Paley assertion

$$L_p(\mathbb{R}^n) = S^0_{p,2}F(\mathbb{R}^n) \tag{1.215}$$

generalises (1.154). If $1 < p < \infty$ and $r \in \mathbb{N}$ then

$$S^r_pW(\mathbb{R}^n) = S^r_{p,2}F(\mathbb{R}^n) \tag{1.216}$$

are the *classical Sobolev spaces with dominating mixed smoothness* which can be equivalently normed by

$$\|f \,|S^r_pW(\mathbb{R}^n)\| \sim \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ 0 \leq \alpha_j \leq r}} \|D^\alpha f \,|L_p(\mathbb{R}^n)\| \tag{1.217}$$

as the natural extension of (1.156) or the obvious counterpart of (1.145).

(iii) *Sobolev spaces.* The lift J_σ in (1.157) is now given by

$$J_\sigma : f \mapsto \Big(\prod_{j=1}^{n} (1 + |\xi_j|^2)^{\sigma/2} \hat{f} \Big)^\vee, \quad \sigma \in \mathbb{R}, \tag{1.218}$$

and, in generalisation of (1.159), the *Sobolev spaces with dominating mixed smoothness* are defined by

$$S^r_p H(\mathbb{R}^n) = J_{-r} L_p(\mathbb{R}^n) = S^r_{p,2} F(\mathbb{R}^n), \quad r \in \mathbb{R},\ 1 < p < \infty. \tag{1.219}$$

As in (1.160) one has

$$S^l_p H(\mathbb{R}^n) = S^l_p W(\mathbb{R}^n), \quad l \in \mathbb{N}_0,\ 1 < p < \infty. \tag{1.220}$$

(iv) *Hölder–Zygmund and Besov spaces*. The mixed differences (1.164) can be naturally extended from two to n dimensions. Then one obtains more or less obvious counterparts of (1.165)–(1.167).

(v) *Atoms*. The rectangles Q_{km} in (1.168) can be naturally extended from two to n dimensions. This applies also to the sequence spaces in Definition 1.40 and the atoms in Definition 1.42. After these modifications one obtains the n dimensional version of Theorem 1.44 for the spaces $S^r_{pq} B(\mathbb{R}^n)$ with unchanged K, L in (1.176) and for the spaces $S^r_{pq} F(\mathbb{R}^n)$ with unchanged K, L in (1.179).

(vi) *Local means*. The kernels W_{km} in (1.183) have natural extensions to n dimensions based on the same one-dimensional functions $w^j_{k_j m_j}$ as there. With appropriately modified sequence spaces in Definition 1.48 one obtains the n dimensional version of Theorem 1.52 for $S^r_{pq} B(\mathbb{R}^n)$ with unchanged A, B as in (1.191) and for $S^r_{pq} F(\mathbb{R}^n)$ with unchanged A, B as in (1.193).

(vii) *Wavelets*. Let $\mathbb{N}^n_{-1}$ be the collection of all $k = (k_1, \ldots, k_n)$ with $k_j \in \mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$. Then the counterpart

$$\{2^{\frac{1}{2}(k_1+\cdots+k_n)} \psi_{km} : k \in \mathbb{N}^n_{-1},\ m \in \mathbb{Z}^n\}, \quad \psi_{km}(x) = \prod_{j=1}^{n} \psi_{k_j m_j}(x_j), \tag{1.221}$$

of (1.198), (1.199) is an orthonormal basis in $L_2(\mathbb{R}^n)$. One has to modify the sequence spaces, quasi-normed by (1.204), (1.205), appropriately. In generalisation of Theorem 1.54 (i) with u as in (1.206) one can characterise $S^r_{pq} B(\mathbb{R}^n)$ by

$$f = \sum_{k \in \mathbb{N}^n_{-1}} \sum_{m \in \mathbb{Z}^n} \lambda_{km}\, 2^{-(k_1+\cdots+k_n)(r-\frac{1}{p})}\, \psi_{km}, \quad \lambda \in s_{pq} b^-, \tag{1.222}$$

with

$$\lambda_{km} = 2^{(k_1+\cdots+k_n)(r-\frac{1}{p}+1)}\, (f, \psi_{km}). \tag{1.223}$$

As there,

$$J: f \mapsto \left\{2^{(k_1+\dots+k_n)(r-\frac{1}{p}+1)}(f, \psi_{km}) : k \in \mathbb{N}^n_{-1},\ m \in \mathbb{Z}^n\right\} \tag{1.224}$$

is an isomorphic map of $S^r_{pq}B(\mathbb{R}^n)$ onto $s_{pq}b^-$. Similarly for $S^r_{pq}F(\mathbb{R}^n)$ with u as in (1.210).

1.2.6 Spaces on domains: definitions, problems

Recall that open sets in $\mathbb{R}^n$ are called domains. Let $g \in S'(\mathbb{R}^n)$. Then the restriction $g|\Omega$ of g to Ω has the same meaning as in (1.86).

Definition 1.56. Let Ω be a domain in $\mathbb{R}^n$, $n \ge 2$ with $\Omega \neq \mathbb{R}^n$. Let $0 < p \le \infty$ ($p < \infty$ for the F-spaces), $0 < q \le \infty$ and $r \in \mathbb{R}$. Let $S^r_{pq}A(\mathbb{R}^n)$ with $A \in \{B, F\}$ be the spaces introduced in Definition 1.38 and extended from $\mathbb{R}^2$ to $\mathbb{R}^n$ as indicated in Section 1.2.5. Then

$$S^r_{pq}A(\Omega) = \left\{f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in S^r_{pq}A(\mathbb{R}^n)\right\}, \tag{1.225}$$

$$\|f\,|S^r_{pq}A(\Omega)\| = \inf \|g\,|S^r_{pq}A(\mathbb{R}^n)\| \tag{1.226}$$

where the infimum is taken over all $g \in S^r_{pq}A(\mathbb{R}^n)$ with $g|\Omega = f$.

Remark 1.57. This is the counterpart of Definition 1.24. In any case $S^r_{pq}A(\Omega)$ is a quasi-Banach space. One may ask for intrinsic characterisations and for linear and bounded extension operators from $S^r_{pq}A(\Omega)$ to $S^r_{pq}A(\mathbb{R}^n)$. In [T08, p. 117] we discussed what could be meant by the somewhat vague notation *intrinsic*. This will not be repeated here in general. But we return later on to a few characterisations which may be called intrinsic.

We discuss in some detail the so-called extension problem. Let

$$\operatorname{re} g = g|\Omega\colon S^r_{pq}A(\mathbb{R}^n) \hookrightarrow S^r_{pq}A(\Omega) \tag{1.227}$$

be the *restriction operator* according to (1.86) as used above. Then one asks whether there is a linear and bounded *extension operator* ext,

$$\operatorname{ext}\colon S^r_{pq}A(\Omega) \hookrightarrow S^r_{pq}A(\mathbb{R}^n), \quad \operatorname{ext} f\,|\Omega = f. \tag{1.228}$$

Using (1.227) this can be written as

$$\operatorname{re} \circ \operatorname{ext} = \operatorname{id} \quad (\text{identity in } S^r_{pq}A(\Omega)). \tag{1.229}$$

This is the counterpart of (1.98)–(1.102). As there one may ask for *common extension operators* or for *universal extension operators*. In case of isotropic spaces we have Theorem 1.27. Furthermore we studied in detail in [T08, Chapter 4] the existence of (common) extension operators for isotropic spaces in rough (bounded) domains, mainly based on wavelet characterisations. One may ask to which extent this method can be applied to spaces with dominating mixed smoothness. We return to this question later on. But there is a decisive difference between isotropic spaces and spaces with dominating mixed smoothness depending heavily on the underlying Cartesian coordinates. Roughly speaking in case of isotropic spaces one is working with atoms and wavelets supported by cubes in Ω or in $\Omega^c = \mathbb{R}^n \setminus \overline{\Omega}$ with side-lengths proportional to the distance to the boundary $\partial\Omega$ of Ω. One can take this rather natural and far-reaching approach to the extension problem for isotropic spaces as a guide. Atoms and wavelets for spaces with dominating mixed smoothness have been discussed in Sections 1.2.2, 1.2.4 with (1.173) as the typical support condition. This suggests to consider domains Ω in $\mathbb{R}^n$ such that for any $x \in \Omega$ there is a cube in Ω of side-length, say, 1, with sides parallel to the axes of coordinates and with x as a corner-point, and/or, for any $y \in \Omega^c$ there is cube in Ω^c of side-length 1 with sides parallel to the axes of coordinates and with y as a corner-point. This is a severe restriction especially if Ω is assumed to be bounded. But it might be of great interest to apply the method developed in [T08, Chapter 4] to spaces with dominating mixed smoothness $S^r_{pq}A(\Omega)$ in domains Ω of the indicated type. Nothing has been done so far. The only paper known to us where conditions of the above type have been used in connection with the extension problem for some spaces with dominating mixed smoothness is [Dsha74]. It relies on integral representations based on derivatives and differences for spaces of type $S^r_{pq}B(\Omega)$. We refer also to [Ull09] where intrinsic characterisations of spaces $S^r_{pq}A(\Omega)$ with $\Omega = \{x \in \mathbb{R}^n, x_j > 0, j = 1, \ldots, l\}$ for $1 \le l \le n$ and linear bounded extensions to $S^r_{pq}A(\mathbb{R}^n)$ are considered. Both [Dsha74], [Ull09] deal with more general spaces.

1.2.7 Spaces on domains: representations, wavelets

Let $S^r_{pq}A(\Omega)$ be the spaces introduced in Definition 1.56 where Ω is now the unit cube in $\mathbb{R}^n$, $n \ge 2$,

$$\mathbb{Q}^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_l < 1\}. \tag{1.230}$$

As before we concentrate on the two-dimensional case, hence $\mathbb{Q}^2$. The generalisation from two dimensions to higher dimensions is a technical matter in the same way as indicated in Section 1.2.5. We deal with representation theorems for the spaces

$$S^r_{pp}B(\mathbb{Q}^2), \quad r \in \mathbb{R},\ 0 < p \le \infty. \tag{1.231}$$

For this purpose we need the wavelet representations for the spaces $S^r_{pp}B(\mathbb{R}^2)$ according to Theorem 1.54 and the corresponding assertion for the spaces $B^r_{pp}(\mathbb{R})$ on the real line $\mathbb{R}$. This is covered by Theorem 1.18 (i) which we now adapt to the notation used in Section 1.2.4. In other words, let

$$\left\{2^{k/2}\psi_{km} : k \in \mathbb{N}_{-1},\ m \in \mathbb{Z}\right\} \tag{1.232}$$

be the same orthonormal basis in $L_2(\mathbb{R})$ as in (1.196), (1.197). Then $f \in S'(\mathbb{R})$ is an element of $B^r_{pp}(\mathbb{R})$ if, and only if, it can be represented as

$$f = \sum_{k=-1}^{\infty} \sum_{m \in \mathbb{Z}} \lambda^k_m \, 2^{-k(r-\frac{1}{p})} \psi_{km}, \quad \lambda \in b^-_{pp}, \tag{1.233}$$

where

$$\lambda = \{\lambda^k_m \in \mathbb{C} : k \in \mathbb{N}_{-1}, \, m \in \mathbb{Z}\} \tag{1.234}$$

and

$$\|f \,|B^r_{pp}(\mathbb{R})\| \sim \|\lambda \,|b^-_{pp}\| = \Big(\sum_{k=-1}^{\infty} \sum_{m \in \mathbb{Z}} |\lambda^k_m|^p \Big)^{1/p}, \quad r \in \mathbb{R}, \; 0 < p \le \infty, \tag{1.235}$$

(usual modification if $p = \infty$). By (1.233) one has

$$\lambda^k_m = \lambda^k_m(f) = 2^{k(r-\frac{1}{p}+1)} \int_{\mathbb{R}} f(t) \, \psi_{km}(t) \, \mathrm{d}t. \tag{1.236}$$

This is a reformulation of Theorem 1.18 (i). Let $I = (0,1)$ be the unit interval on the real line $\mathbb{R}$ and let $B^r_{pp}(I)$ be the restriction of $B^r_{pp}(\mathbb{R})$ to I according to Definition 1.24. Let $S^r_{pp}B(\mathbb{R}^2)$ be as in Definition 1.38 and let

$$S^r_{pp}B(I^2), \quad I^2 = \mathbb{R} \times I = \{x = (x_1, x_2) \in \mathbb{R}^2, \; x_1 \in \mathbb{R}, \; 0 < x_2 < 1\} \tag{1.237}$$

be the corresponding spaces with dominating mixed smoothness as introduced in Definition 1.56 with respect to the strip I^2.

Theorem 1.58. *Let $0 < p \le \infty$ and $r \in \mathbb{R}$. Let*

$$\{\psi_{jm} : j \in \mathbb{N}_{-1}, \, m \in \mathbb{Z}\} \tag{1.238}$$

be an orthogonal Daubechies wavelet basis in $L_2(\mathbb{R})$ according to (1.196) (with sufficiently many moment conditions for ψ_M).

(i) *Let $f \in S'(\mathbb{R}^2)$. Then $f \in S^r_{pp}B(\mathbb{R}^2)$ if, and only if, it can be represented as*

$$f(x_1, x_2) = \sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} f_{jm}(x_2) \, 2^{-j(r-\frac{1}{p})} \psi_{jm}(x_1) \tag{1.239}$$

with

$$\big\| \{f_{jm}\} \,|\ell_p\big(B^r_{pp}(\mathbb{R})\big) \big\| = \Big(\sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} \|f_{jm} \,|B^r_{pp}(\mathbb{R})\|^p \Big)^{1/p} < \infty, \tag{1.240}$$

unconditional convergence being locally in any space $S^\varrho_{pp}B(\mathbb{R}^2)$ with $\varrho < r$. If $f \in S^r_{pp}B(\mathbb{R}^2)$ then the representation (1.239) is unique and

$$f_{jm}(x_2) = 2^{j(r-\frac{1}{p}+1)} \int_{\mathbb{R}} f(x_1, x_2) \, \psi_{jm}(x_1) \, \mathrm{d}x_1, \tag{1.241}$$

$x_2 \in \mathbb{R}$ (*appropriately interpreted*). *Furthermore,*

$$\|f\,|S^r_{pp}B(\mathbb{R}^2)\| \sim \big\|\{f_{jm}\}\,|\ell_p\big(B^r_{pp}(\mathbb{R})\big)\big\|, \tag{1.242}$$

equivalent quasi-norms.

(ii) *Let* $f \in D'(I^2)$. *Then* $f \in S^r_{pp}B(I^2)$ *if, and only if, it can be represented by* (1.239) *where now* $x = (x_1, x_2) \in I^2$ *with*

$$\big\|\{f_{jm}\}\,|\ell_p\big(B^r_{pp}(I)\big)\big\| = \Big(\sum_{j=-1}^{\infty}\sum_{m\in\mathbb{Z}} \|f_{jm}\,|B^r_{pp}(I)\|^p\Big)^{1/p} < \infty, \tag{1.243}$$

unconditional convergence being locally in any space $S^{\varrho}_{pp}B(I^2)$ *with* $\varrho < r$. *If* $f \in S^r_{pp}B(I^2)$ *then the representation* (1.239) *with* $(x_1, x_2) \in I^2$ *is unique with* (1.241) *where now* $x_2 \in I$. *Furthermore,*

$$\|f\,|S^r_{pp}B(I^2)\| \sim \big\|\{f_{jm}\}\,|\ell_p\big(B^r_{pp}(I)\big)\big\|, \tag{1.244}$$

equivalent quasi-norms.

Proof. *Step* 1. For the proof of part (i) we use Theorem 1.54 (i). By (1.207) and (1.204) with $p = q$ we have

$$f = \sum_{k_1=-1}^{\infty}\sum_{m_1\in\mathbb{Z}} 2^{-k_1(r-\frac{1}{p})}\,\psi_{k_1m_1}(x_1)\,f_{k_1m_1}(x_2) \tag{1.245}$$

with

$$f_{k_1m_1}(x_2) = \sum_{k_2=-1}^{\infty}\sum_{m_2\in\mathbb{Z}} \lambda_{(k_1,k_2),(m_1,m_2)} 2^{-k_2(r-\frac{1}{p})}\psi_{k_2m_2}(x_2) \tag{1.246}$$

and, according to (1.233)–(1.235),

$$\|f_{k_1m_1}\,|B^r_{pp}(\mathbb{R})\| \sim \Big(\sum_{k_2,m_2} |\lambda_{(k_1,k_2),(m_1,m_2)}|^p\Big)^{1/p} \tag{1.247}$$

(modification if $p = \infty$). Then it follows from the isomorphic map (1.208), (1.209) that

$$\|f\,|S^r_{pp}B(\mathbb{R}^2)\| \sim \|\lambda\,|s_{pp}b^-\| \sim \|\{f_{jm}\}\,|\ell_p\big(B^r_{pp}(\mathbb{R})\big)\|, \tag{1.248}$$

hence (1.242). Finally one obtains (1.241) from the orthonormalisation in (1.232).

Step 2. We prove part (ii). Let $f \in D'(I^2)$ be the restriction both of $g^1 \in S^r_{pp}B(\mathbb{R}^2)$ and $g^2 \in S^r_{pp}B(\mathbb{R}^2)$ to I^2. Let

$$g^l = \sum_{j=-1}^{\infty}\sum_{m\in\mathbb{Z}} g^l_{jm}(x_2)\,2^{-j(r-\frac{1}{p})}\,\psi_{jm}(x_1), \quad l = 1, 2, \tag{1.249}$$

with the counterpart of (1.240)–(1.242). Let $\varphi(x) = \varphi_1(x_1)\varphi_2(x_2)$ with $\varphi_1 \in D(\mathbb{R})$ and $\varphi_2 \in D(I)$. Then $g^1(\varphi) = g^2(\varphi)$ and, hence,

$$\sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} \big(g^1_{jm} - g^2_{jm}, \varphi_2\big)\, 2^{-j(r-\frac{1}{p})} \big(\psi_{jm}, \varphi_1\big) = 0. \tag{1.250}$$

In Remark 1.19 we referred to the technicalities resulting in Theorem 1.18 and also in Theorem 1.54 (ensuring convergence based on duality). In particular one may choose a sequence of functions $\varphi_1 \in D(\mathbb{R})$ converging to ψ_{jm} for fixed j and m. Then it follows from (1.250) that

$$\big(g^1_{jm}, \varphi_2\big) = \big(g^2_{jm}, \varphi_2\big), \quad \varphi_2 \in D(I). \tag{1.251}$$

Conversely, if one has (1.251) for $j \in \mathbb{N}_{-1}$, $m \in \mathbb{Z}$, then it follows by standard arguments that

$$(g^1, \varphi) = (g^2, \varphi), \quad \varphi \in D(I^2), \tag{1.252}$$

(approximation of $\varphi \in D(I^2)$ by finite linear combinations of products of functions $\varphi_1 \in D(\mathbb{R})$ and $\varphi_2 \in D(I)$). In other words,

$$f = g^1|I^2 = g^2|I^2 \quad \text{if, and only if,} \quad g^1_{jm}|I = g^2_{jm}|I, \tag{1.253}$$

for all $j \in \mathbb{N}_{-1}$ and $m \in \mathbb{Z}$. Let now $f \in S^r_{pp}B(I^2)$ be the restriction of $g \in S^r_{pp}B(\mathbb{R}^2)$ represented by the g-counterpart of (1.239) with (1.242). Let $f_{jm} = g_{jm}|I^2$. Then it follows from the above considerations that f can be represented by (1.239) where $x_2 \in I$ with (1.243), (1.244). □

Remark 1.59. Part (i) can be reformulated in terms of vector-valued Besov spaces $B^r_{pp}(\mathbb{R}, X)$ with $X = B^r_{pp}(\mathbb{R})$ or in terms of tensor products $B^r_{pp}(\mathbb{R}) \otimes B^r_{pp}(\mathbb{R})$. We preferred the above direct approach. But we wish to refer to the survey [Schm87] as far as vector-valued Sobolev and Besov spaces are concerned and to the recent paper [SiU08] dealing with tensor products of related spaces employing wavelet representations.

In Chapters 2, 3 we use part (ii) of the above theorem to study Haar bases and Faber bases for spaces with dominating mixed smoothness in cubes $\mathbb{Q}^n$ according to (1.230). But first we apply the method developed above to construct wavelet bases in some spaces $S^r_{pp}B(\mathbb{Q}^2)$. We rely on [T08] and recall some assertions obtained there. Let again $I = (0,1)$ be the unit interval in $\mathbb{R}$. We adapt the definitions in [T08, Section 6.1.1, pp. 178–180] to the above situation. Let

$$\mathbb{Z}^I = \big\{x^j_l \in I : j \in \mathbb{N}_0;\ l = 1, \ldots, N_j\big\}, \quad N_j \in \mathbb{N}, \tag{1.254}$$

typically with $N_j \sim 2^j$, such that for some $c_1 > 0$,

$$|x^j_l - x^j_{l'}| \ge c_1\, 2^{-j}, \quad j \in \mathbb{N}_0,\ l \ne l'. \tag{1.255}$$

Let $0 < p \le \infty$. Then $b_{pp}(\mathbb{Z}^I)$ is the collection of all sequences

$$\lambda = \{\lambda_l^j \in \mathbb{C} : j \in \mathbb{N}_0;\ l = 1, \ldots, N_j\} \tag{1.256}$$

such that

$$\|\lambda\,|b_{pp}(\mathbb{Z}^I)\| = \Big(\sum_{j=0}^{\infty}\sum_{l=1}^{N_j} |\lambda_l^j|^p\Big)^{1/p} < \infty \tag{1.257}$$

(obviously modified if $p = \infty$). Of course, $b_{pp}(\mathbb{Z}^I)$ is a quasi-Banach space, quasi-normed by (1.257). Let

$$B(x,t) = \{y \in \mathbb{R} : |y - x| < t\}, \quad x \in \mathbb{R},\ t > 0. \tag{1.258}$$

Recall that $C^u(I)$ with $u \in \mathbb{N}$ is the collection of all complex-valued continuously differentiable functions up to order u inclusively in the closed interval $\bar{I} = [0,1]$. Let $\Gamma = \{0,1\}$ be the boundary of I consisting of the points 0 and 1. Let $\psi^{(k)} = \frac{\mathrm{d}^k\psi}{\mathrm{d}x^k}$ be the usual classical derivatives.

Definition 1.60. Let again $I = (0,1)$ be the unit interval in $\mathbb{R}$ and let $\mathbb{Z}^I$ be as in (1.254), (1.255). Let $u \in \mathbb{N}$.

(i) Then

$$\Psi^I = \{\psi_{jl} : j \in \mathbb{N}_0;\ l = 1, \ldots, N_j\} \subset C^u(I) \tag{1.259}$$

is called a *u-wavelet system* in I if for some $c_2 > 0$, $c_3 > 0$,

$$\operatorname{supp} \psi_{jl} \subset B\big(x_l^j, c_2 2^{-j}\big) \cap [0,1], \quad j \in \mathbb{N}_0;\ l = 1, \ldots, N_j, \tag{1.260}$$

and

$$|\psi_{jl}^{(k)}(x)| \le c_3\, 2^{jk}, \quad j \in \mathbb{N}_0;\ l = 1, \ldots, N_j;\ x \in I, \tag{1.261}$$

for $k = 0, \ldots, u$.

(ii) The above u-wavelet system Ψ^I is called *interior* if, in addition, for some $c_4 > 0$,

$$\operatorname{dist}\big(B\big(x_l^j, c_2\, 2^{-j}\big), \Gamma\big) \ge c_4\, 2^{-j}, \quad j \in \mathbb{N}_0;\ l = 1, \ldots, N_j. \tag{1.262}$$

Remark 1.61. This is a simplified version of [T08, Definitions 5.25, 6.3, pp. 152, 179]. In connection with the extension problem some cancellations of the above wavelets might be useful. A corresponding description will be given when needed. Recall again that $B^r_{pp}(I)$ is the restriction of $B^r_{pp}(\mathbb{R})$ to I according to Definition 1.24.

Theorem 1.62. *Let again $I = (0,1)$ be the unit interval on $\mathbb{R}$ and let $B^r_{pp}(I)$ with $r \in \mathbb{R}$ and $0 < p \le \infty$ be the above Besov spaces.*

(i) *Let $u \in \mathbb{N}$. Then there is a common orthogonal interior u-wavelet system $\Psi^I = \{\psi_{jl}\}$ according to Definition 1.60* (ii) *for all p, r with*

$$\begin{cases} 0 < p \le \infty, & \max\big(\frac{1}{p}, 1\big) - 1 < r < \frac{1}{p}, \\ 1 < p < \infty, & r = 0, \\ 0 < p \le \infty, & r < 0, \end{cases} \tag{1.263}$$

and

$$u > \max\big(r,\ \max\big(\tfrac{1}{p},1\big) - 1 - r\big), \tag{1.264}$$

such that $f \in D'(I)$ *belongs to* $B^r_{pp}(I)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{l=1}^{N_j} \lambda^j_l \, 2^{-j(r-\frac{1}{p})} \, \psi_{jl}, \quad \lambda \in b_{pp}(\mathbb{Z}^I), \tag{1.265}$$

unconditional convergence being in $D'(I)$ *and in* $B^{\varrho}_{pp}(I)$ *for any* $\varrho < r$. *If* $f \in B^r_{pp}(I)$ *then the representation* (1.265) *is unique,* $\lambda = \lambda(f)$, *with*

$$\lambda^j_l(f) = 2^{j(r-\frac{1}{p}+1)} \int_I f(x)\, \psi_{jl}(x)\, \mathrm{d}x \tag{1.266}$$

(*appropriately interpreted*) *and*

$$J: \ f \in B^r_{pp}(I) \mapsto \lambda(f) \tag{1.267}$$

is an isomorphic map of $B^r_{pp}(I)$ *onto* $b_{pp}(\mathbb{Z}^I)$. *If, in addition,* $p < \infty$ *then* Ψ^I *is an unconditional basis in* $B^r_{pp}(I)$ (*common orthogonal interior* u*-wavelet basis*).

(ii) *Let* $u \in \mathbb{N}$, $v \in \mathbb{N}$ *with* $v < u$. *Then there is a common* u*-wavelet system* $\Psi^I = \{\psi_{jl}\}$ *according to Definition* 1.60 (i) *for all* p, r *with*

$$1 \le p < \infty, \quad \frac{1}{p} - 1 < r - v < \frac{1}{p}, \tag{1.268}$$

such that $f \in D'(I)$ *belongs to* $B^r_{pp}(I)$ *if, and only if, it can be represented by* (1.265), *unconditional convergence being in* $B^r_{pp}(I)$. *If* $f \in B^r_{pp}(I)$ *then the representation is unique,* $\lambda = \lambda(f)$, *and* J *in* (1.267) *is an isomorphic map of* $B^r_{pp}(I)$ *onto* $b_{pp}(\mathbb{Z}^I)$ (*common* u*-wavelet basis*).

Remark 1.63. This theorem is an adapted version of more general assertions in [T08, Theorems 3.13, 3.23, 5.35, 6.7, pp. 80/81, 89/90, 162, 181]. Compared with the formulations given there we used in addition that

$$B^r_{pp}(I) = \widetilde{B}^r_{pp}(I), \quad 0 < p \le \infty, \ \frac{1}{p} - 1 < r < \frac{1}{p}, \tag{1.269}$$

where $\widetilde{B}^r_{pp}(I)$ is a special case of Definition 1.24 and (1.92). This is a well-known pointwise multiplier assertion mentioned in [T08, Proposition 6.12, p. 182]. One may also consult [T01, p. 58]. In (1.266) we assumed that $\big\{2^{j/2}\psi_{jl}\big\}$ is L_2-orthonormal.

After this preparation we return to the spaces in (1.231) asking for wavelet representations. On the one hand the situation is similar as in Theorem 1.54. On the other hand our method, a combination of Theorems 1.58, 1.62, works only for spaces $S^r_{pq}B(\mathbb{Q}^2)$ with $p = q$, although the outcome should be valid on a larger scale. Using other arguments we construct later on Haar bases and Faber bases also for some spaces $S^r_{pq}B(\mathbb{Q}^2)$ with $p \neq q$. First we need the counterparts of the sequence spaces $b_{pq}(\mathbb{Z}^I)$ in (1.257) and of the wavelet system Ψ^I in (1.259) with $\mathbb{Q}^2$ according to (1.230) in place of I. We put temporarily $Q = \mathbb{Q}^2$.

Definition 1.64. Let $\Psi = \Psi^I = \{\psi_{jl}\}$ be an (interior) u-wavelet system according to Definition 1.60, $u \in \mathbb{N}$. Let

$$\psi_{km}(x) = \psi_{k_1 m_1}(x_1)\, \psi_{k_2 m_2}(x_2), \quad k \in \mathbb{N}_0^2,\ m_l = 1, \ldots, N_{k_l}, \tag{1.270}$$

$x \in Q$. Let

$$\mathbb{P}^\Psi_k = \big\{ m \in \mathbb{Z}^2 \text{ with } m \text{ as in (1.270)} \big\}, \quad k \in \mathbb{N}_0^2, \tag{1.271}$$

and

$$\mathbb{Z}^Q = \mathbb{Z}^I \times \mathbb{Z}^I = \big\{ (x^{k_1}_{m_1}, x^{k_2}_{m_2}) : k \in \mathbb{N}_0^2,\ m \in \mathbb{P}^\Psi_k \big\}. \tag{1.272}$$

Then

$$\Psi^Q = \Psi^I \times \Psi^I = \big\{ \psi_{km} : k \in \mathbb{N}_0^2,\ m \in \mathbb{P}^\Psi_k \big\} \tag{1.273}$$

is called an (interior) u-wavelet system in Q. Let $0 < p \le \infty$. Then $s_{pp}b(\mathbb{Z}^Q)$ is the quasi-Banach space of all sequences

$$\mu = \big\{ \mu_{km} : k \in \mathbb{N}_0^2,\ m \in \mathbb{P}^\Psi_k \big\} \tag{1.274}$$

quasi-normed by

$$\|\mu \,|s_{pp}b(\mathbb{Z}^Q)\| = \Big(\sum_{k \in \mathbb{N}_0^2} \sum_{m \in \mathbb{P}^\Psi_k} |\mu_{km}|^p \Big)^{1/p} \tag{1.275}$$

(with the usual modification if $p = \infty$).

Remark 1.65. We try to simplify our notation as much as possible, also at the expense that some symbols are used in different meanings. If extra clarity is needed then the Daubechies wavelets ψ_{jm} in Theorem 1.58 are denoted temporarily by ψ^D_{jm}, whereas ψ^I_{jm} stands for the wavelets in Definition 1.60 and Theorem 1.62. In the proof of the following theorem we need both types of wavelets. Then, and only then, we use the indicated additional marks.

Theorem 1.66. *Let $Q = \mathbb{Q}^2$ be the unit square in $\mathbb{R}^2$ according to (1.230) with $n = 2$. Let $S^r_{pp}B(Q)$ be the corresponding spaces with dominating mixed smoothness as introduced in Definition 1.56 and (1.231).*

(i) *Let $u \in \mathbb{N}$ and p, r be as in (1.263), (1.264). Let $\Psi^Q = \Psi^I \times \Psi^I$ be as in (1.273) where $\Psi^I = \{\psi_{jl}\} = \{\psi^I_{jl}\}$ is the common orthogonal interior u-wavelet basis according to Theorem 1.62 (i). Let $f \in D'(Q)$. Then $f \in S^r_{pp}B(Q)$ if, and only if, it can be represented as*

$$f = \sum_{k \in \mathbb{N}_0^2} \sum_{m \in \mathbb{P}^\Psi_k} \lambda_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})}\, \psi_{km}, \quad \lambda \in s_{pp}b(\mathbb{Z}^Q), \tag{1.276}$$

unconditional convergence being in $D'(Q)$ *and in* $S^{\varrho}_{pp}B(Q)$ *for any* $\varrho < r$. *If* $f \in S^r_{pp}B(Q)$ *then the representation* (1.276) *is unique ,* $\lambda = \lambda(f)$, *with*

$$\lambda_{km}(f) = 2^{(k_1+k_2)(r-\frac{1}{p}+1)} \int_Q f(x)\,\psi_{km}(x)\,\mathrm{d}x \tag{1.277}$$

(*appropriately interpreted*) *and*

$$J : f \in S^r_{pp}B(Q) \mapsto \lambda(f) \tag{1.278}$$

is an isomorphic map of $S^r_{pp}B(Q)$ *onto* $s_{pp}b(\mathbb{Z}^Q)$. *If, in addition,* $p < \infty$ *then* Ψ^Q *is an unconditional basis in* $S^r_{pp}B(Q)$.

(ii) *Let* $u \in \mathbb{N}$, $v \in \mathbb{N}$ *with* $v < u$, *and let* p, r *be as in* (1.268). *Let* $\Psi^Q = \Psi^I \times \Psi^I$ *be as in* (1.273) *where now* $\Psi^I = \{\psi_{jl}\} = \{\psi^I_{jl}\}$ *is the common u-wavelet basis according to Theorem* 1.62 (ii). *Let* $f \in D'(Q)$. *Then* $f \in S^r_{pp}B(Q)$ *if, and only if, it can be represented in terms of this u-wavelet basis by* (1.276), *unconditional convergence being in* $S^r_{pp}B(Q)$. *If* $f \in S^r_{pp}B(Q)$ *then this representation is unique,* $\lambda = \lambda(f)$. *Furthermore,* J *in* (1.278) *is an isomorphic map onto* $s_{pp}b(\mathbb{Z}^Q)$ *and* Ψ^Q *is a common unconditional basis in* $S^r_{pp}B(Q)$.

Proof. Step 1. We prove part (i). As indicated in Remark 1.65 we distinguish now between the Daubechies wavelets ψ^D_{jl} and the wavelets ψ^I_{jl} from Theorem 1.62 (i). We use Theorem 1.58 (ii). In particular $f \in S^r_{pp}B(I^2)$ can be represented by

$$f(x_1, x_2) = \sum_{k_1=-1}^{\infty} \sum_{m_1 \in \mathbb{Z}} f_{k_1 m_1}(x_2)\, 2^{-k_1(r-\frac{1}{p})}\, \psi^D_{k_1 m_1}(x_1), \tag{1.279}$$

$x_1 \in \mathbb{R}$, $x_2 \in I$, with $f_{k_1 m_1} \in B^r_{pp}(I)$ and

$$\| f \,| S^r_{pp}B(I^2)\| \sim \Big(\sum_{k_1=-1}^{\infty} \sum_{m_1 \in \mathbb{Z}} \| f_{k_1 m_1} \,| B^r_{pp}(I)\|^p \Big)^{1/p}. \tag{1.280}$$

We expand each $f_{k_1 m_1} \in B^r_{pp}(I)$ according to Theorem 1.62 (i),

$$f_{k_1 m_1}(x_2) = \sum_{k_2=0}^{\infty} \sum_{m_2=1}^{N_{k_2}} \lambda^{k_1,k_2}_{m_1,m_2}\, 2^{-k_2(r-\frac{1}{p})}\, \psi^I_{k_2 m_2}(x_2). \tag{1.281}$$

We insert this representation in (1.279) and obtain

$$\begin{aligned} f(x_1, x_2) &= \sum_{k_1=-1}^{\infty} \sum_{k_2=0}^{\infty} \sum_{m_1 \in \mathbb{Z}} \sum_{m_2=1}^{N_{k_2}} \lambda^{k_1,k_2}_{m_1,m_2}\, 2^{-(k_1+k_2)(r-\frac{1}{p})}\, \psi^D_{k_1 m_1}(x_1)\, \psi^I_{k_2 m_2}(x_2) \\ &= \sum_{k_2=0}^{\infty} \sum_{m_2=1}^{N_{k_2}} 2^{-k_2(r-\frac{1}{p})} \psi^I_{k_2 m_2}(x_2)\, g_{k_2 m_2}(x_1) \end{aligned} \tag{1.282}$$

with

$$g_{k_2m_2}(x_1) = \sum_{k_1=-1}^{\infty} \sum_{m_1\in\mathbb{Z}} \lambda^{k_1,k_2}_{m_1,m_2}\, 2^{-k_1(r-\frac{1}{p})}\, \psi^D_{k_1m_1}(x_1) \tag{1.283}$$

and as in (1.246), (1.247) with a reference to (1.235)

$$\|g_{k_2m_2}\,|B^r_{pp}(\mathbb{R})\| \sim \Big(\sum_{k_1=-1}^{\infty} \sum_{m_1\in\mathbb{Z}} |\lambda^{k_1,k_2}_{m_1,m_2}|^p \Big)^{1/p}. \tag{1.284}$$

By (1.280) and (1.267) one has

$$\begin{aligned} \|f\,|S^r_{pp}B(I^2)\| &\sim \Big(\sum_{k_1=-1}^{\infty} \sum_{k_2=0}^{\infty} \sum_{m_1\in\mathbb{Z}} \sum_{m_2=1}^{N_{k_2}} |\lambda^{k_1,k_2}_{m_1,m_2}|^p \Big)^{1/p} \\ &\sim \Big(\sum_{k_2=0}^{\infty} \sum_{m_2=1}^{N_{k_2}} \|g_{k_2m_2}\,|B^r_{pp}(\mathbb{R})\|^p \Big)^{1/p}. \end{aligned} \tag{1.285}$$

Obviously, $S^r_{pp}B(Q)$ can also be obtained in the usual way as the restriction of $S^r_{pp}B(I^2)$ to Q. But then we have the same situation as in Step 2 of the proof of Theorem 1.58 where x_1 and x_2 change their roles. Using again Theorem 1.62 (i) now with respect to x_1 one obtains the representation (1.276) and

$$\|f\,|S^r_{pp}B(Q)\| \sim \|\lambda\,|s_{pp}b(\mathbb{Z}^Q)\|. \tag{1.286}$$

Conversely, if f is given by (1.276) then one can argue backwards using that (1.267) is a map onto, first from Q to I^2 and then from I^2 to $\mathbb{R}^2$. Finally for the orthogonal basis $\{\psi_{km}\}$ the assertion (1.277) follows from (1.266).

Step 2. The proof of part (ii) is the same using now Theorem 1.62 (ii). □

1.2.8 Spaces on domains: extensions, intrinsic characterisations

Whether one accepts the wavelet expansions in Theorem 1.66 as intrinsic is a matter of taste, even for the orthogonal bases as in (1.276), (1.277). But in any case one would ask for more explicit descriptions in terms of derivatives and differences. For the Sobolev spaces $S^r_pW(\mathbb{R}^2)$ one has the equivalent norms (1.145), (1.156) and for the Besov spaces $S^r_{pq}B(\mathbb{R}^2)$ the equivalent norms (1.166), (1.167). Are there counterparts on domains with $\mathbb{Q}^2$ as the case of preference? For isotropic Sobolev spaces $W^k_p(\Omega)$ and Besov spaces $B^s_{pq}(\Omega)$ in, say, bounded Lipschitz domains Ω in $\mathbb{R}^n$ one has rather satisfactory assertions which we briefly described at the end of Section 1.1.6.

First we collect some known assertions. Let again $I = (0,1)$ be the unit interval on the real line $\mathbb{R}$ and let for $M \in \mathbb{N}$,

$$(\Delta^M_{h,I} f)(x) = \begin{cases} (\Delta^M_h f)(x) & \text{if } x + lh \in I \text{ for } l = 0,\ldots,M, \\ 0 & \text{otherwise,} \end{cases} \tag{1.287}$$

be the same differences adapted to I as in (1.106). If

$$0 < p, q \le \infty, \quad \max\left(\tfrac{1}{p}, 1\right) - 1 < r < M \in \mathbb{N}, \tag{1.288}$$

then $B^r_{pq}(I)$ is the collection of all $f \in L_1(I)$ such that

$$\|f\,|L_p(I)\| + \left(\int_0^1 t^{-rq} \sup_{0<h<t} \big\|\Delta^M_{h,I} f\,|L_p(I)\big\|^q \frac{dt}{t}\right)^{1/q} < \infty \tag{1.289}$$

(equivalent quasi-norms). This is a special case of (1.108) and the references given there. By the arguments in [T83, p. 112] one can replace (1.289) by

$$\|f\,|B^r_{pq}(I)\|_M = \|f\,|L_p(I)\| + \left(\int_0^1 h^{-rq} \big\|\Delta^M_{h,I} f\,|L_p(I)\big\|^q \frac{dh}{h}\right)^{1/q} < \infty \tag{1.290}$$

(equivalent quasi-norms). In (1.166) we described equivalent norms in the spaces $S^r_{pq}B(\mathbb{R}^2)$ with $1 \le p, q \le \infty$, $r > 0$. As mentioned there one can extend these assertions to some $p < 1$ and/or $q < 1$, but not in a satisfactory way. One may ask whether there are counterparts for the spaces $S^r_{pq}B(\mathbb{Q}^2)$ with $1 \le p, q \le \infty$, $r > 0$. We do not have a final answer but we can say something if $p = q$. First we introduce some notation in the unit square $Q = \mathbb{Q}^2$ according to (1.230) combining (1.163), (1.164) with (1.287). Let $(\Delta^M_{h,1} f)(x)$ with $x \in \mathbb{R}^2$ be the differences with respect to the x_1-direction according to (1.163), $M \in \mathbb{N}$. Then

$$\Delta^M_{h,1,Q} f(x) = \begin{cases} \Delta^M_{h,1} f(x) & \text{if } (x_1 + lh, x_2) \in Q \text{ for } l = 0, \dots, M, \\ 0 & \text{otherwise.} \end{cases} \tag{1.291}$$

Similarly $\Delta^M_{h,2,Q} f$. Let

$$\Delta^{M,M}_h f(x) = \Delta^{M,M}_{h_1,h_2} f(x) = \Delta^M_{h_2,2}\big(\Delta^M_{h_1,1} f\big)(x), \quad x \in \mathbb{R}^2, \tag{1.292}$$

as in (1.164) and

$$\Delta^{M,M}_{h,Q} f(x) = \begin{cases} \Delta^{M,M}_h f(x) & \text{if } (x_1 + l_1 h_1, x_2 + l_2 h_2) \in Q,\ l_m = 0, \dots, M, \\ 0 & \text{otherwise.} \end{cases} \tag{1.293}$$

Let $1 \le p \le \infty$, $0 < r < M \in \mathbb{N}$. We put

$$\begin{aligned} &\|f\,|S^r_{pp}B(Q)\|^M \\ &\quad = \|f\,|L_p(Q)\| \\ &\qquad + \left(\int_0^1 h^{-rp}\Big(\big\|\Delta^M_{h,1,Q} f\,|L_p(Q)\big\|^p + \big\|\Delta^M_{h,2,Q} f\,|L_p(Q)\big\|^p\Big) \frac{dh}{h}\right)^{1/p} \\ &\qquad + \left(\int_0^1 \int_0^1 (h_1 h_2)^{-rp} \big\|\Delta^{M,M}_{h,Q} f\,|L_p(Q)\big\|^p \frac{dh}{h_1 h_2}\right)^{1/p}. \end{aligned} \tag{1.294}$$

Using (1.290) with $p = q$ one has

$$\|f\,|S^r_{pp}B(Q)\|^M \sim \Big(\int_0^1 \|f(\cdot,x_2)\,|B^r_{pp}(I)\|^p_M \,\mathrm{d}x_2\Big)^{1/p} + \Big(\int_0^1\int_0^1 h^{-rp}\,\|\Delta^M_{h,2,Q}f(\cdot,x_2)\,|B^r_{pp}(I)\|^p_M \,\frac{\mathrm{d}h}{h}\,\mathrm{d}x_2\Big)^{1/p}. \tag{1.295}$$

Here $\|\cdot|B^r_{pp}(I)\|$ is applied with respect to the x_1-direction. This is the counterpart of (1.166). Recall that $S^r_pW(\mathbb{R}^2)$, $1 < p < \infty$, $r \in \mathbb{N}$, with (1.155) and the equivalent norms (1.145), (1.156) are the classical Sobolev spaces with dominating mixed smoothness. Let $S^r_pW(\mathbb{Q}^2) = S^r_{p,2}F(\mathbb{Q}^2)$ according to Definition 1.56 be the restriction to $\mathbb{Q}^2$. Recall that

$$\|f\,|W^r_p(I)\| = \|f\,|L_p(I)\| + \Big\|\frac{\mathrm{d}^r f}{\mathrm{d}x^r}\,|L_p(I)\Big\| \tag{1.296}$$

is an equivalent norm for $W^r_p(I)$, $1 < p < \infty$, $r \in \mathbb{N}$. With $Q = \mathbb{Q}^2$ let

$$\begin{aligned}\|f\,|S^r_pW(Q)\|^* &= \|f\,|L_p(Q)\| + \Big\|\frac{\partial^r f}{\partial x_1^r}\,|L_p(Q)\Big\| + \Big\|\frac{\partial^r f}{\partial x_2^r}\,|L_p(Q)\Big\| + \Big\|\frac{\partial^{2r} f}{\partial x_1^r\partial x_2^r}\,|L_p(Q)\Big\| \\ &\sim \Big(\int_0^1 \|f(\cdot,x_2)\,|W^r_p(I)\|^p\,\mathrm{d}x_2\Big)^{1/p} + \Big(\int_0^1 \Big\|\frac{\partial^r f}{\partial x_2^r}(\cdot,x_2)\,|W^r_p(I)\Big\|^p\,\mathrm{d}x_2\Big)^{1/p}\end{aligned} \tag{1.297}$$

be the counterpart of (1.294), (1.295). In bounded Lipschitz domains Ω in $\mathbb{R}^n$ one has for some isotropic spaces $B^s_{pq}(\Omega)$ and $W^k_p(\Omega)$ the intrinsic quasi-norms (1.104), (1.108) and also the extension property described in Theorem 1.27. We ask now for counterparts for the above spaces $S^r_{pp}B(Q)$ and $S^r_pW(Q)$. We explained in Section 1.2.6 what is meant by a (common) extension operator. Let $S^0_pW(Q) = L_p(Q)$.

Theorem 1.67. *Let $Q = \mathbb{Q}^2$ be the unit square in $\mathbb{R}^2$ according to (1.230) with $n = 2$.*

(i) *Let $1 \le p \le \infty$, $M \in \mathbb{N}$, and $0 < r < M$. Let $S^r_{pp}B(Q)$ be the corresponding Besov spaces with dominating mixed smoothness as introduced in Definition 1.56 and (1.231). Then there exists a common extension operator* ext_M,

$$\mathrm{ext}_M : S^r_{pp}B(Q) \hookrightarrow S^r_{pp}B(\mathbb{R}^2). \tag{1.298}$$

Furthermore, $S^r_{pp}B(Q)$ is the collection of all $f \in L_1(Q)$ (or likewise $f \in L_p(Q)$) such that

$$\|f\,|S^r_{pp}B(Q)\|^M < \infty \quad \text{according to (1.294)} \tag{1.299}$$

(equivalent norms).

(ii) *Let $1 < p < \infty$, $M \in \mathbb{N}$, and $r \in \mathbb{N}_0$ with $r \le M$. Let $S^r_pW(Q) = S^r_{p,2}F(Q)$ be the above Sobolev spaces with dominating mixed smoothness. Then the extension operator* ext_M *in (1.298) is also a common extension operator,*

$$\mathrm{ext}_M : S^r_pW(Q) \hookrightarrow S^r_pW(\mathbb{R}^2). \tag{1.300}$$

Furthermore, $S^r_p W(Q)$ is the collection of all $f \in L_1(Q)$ (or likewise $f \in L_p(Q)$) such that

$$\| f \,| S^r_p W(Q)\|^* < \infty \quad \textit{according to (1.297)} \tag{1.301}$$

(equivalent norms).

Proof. Step 1. By (1.145), (1.166) with $p = q$ and Definition 1.56 it follows that

$$\| f \,| S^r_p W(Q)\|^* \le c \, \| f \,| S^r_p W(Q)\| \tag{1.302}$$

and

$$\| f \,| S^r_{pp} B(Q)\|^M \le c \, \| f \,| S^r_{pp} B(Q)\|. \tag{1.303}$$

Step 2. It is well known that the classical extension method by Hestenes, [Hes41], works for all extensions of $A^s_{pq}(\mathbb{R}^n_+)$ to $A^s_{pq}(\mathbb{R}^n)$ with $1 \le p, q \le \infty$, $s > 0$, [T92, Section 4.5.2, p. 223], $n \ge 1$. One may also consult [HaT08], where we described in Section 4.6.1, p. 112, the history of this method. With an additional cut-off this can also be used for the above spaces $W^r_p(I)$ and $B^r_{pp}(I)$. We apply this observation for frozen $x_2 \in I$ to $f(\cdot, x_2)$ and $\frac{\partial^r f}{\partial x_2^r}(\cdot, x_2)$ in the last equivalence of (1.297) with respect to $W^r_p(I)$. This gives the desired extension from Q to I^2. Applying this method again (now for frozen x_1 and with respect to x_2) one obtains the desired extension to $\mathbb{R}^2$, hence (1.300) and also (1.301) as a characterisation (equivalent norm). Using (1.295) one can apply this argument also to the spaces $S^r_{pp} B(Q)$ with the same extension operator as in (1.300). This proves (1.298) and also (1.299) as a characterisation (equivalent norms). □

The spaces $S^1_p W(\mathbb{Q}^n)$, $1 < p < \infty$, will play a crucial role later on. We need a few additional properties, traces on $\partial \mathbb{Q}^n$ and equivalent norms, which are well known. But it seems to be reasonable to give direct short proofs. Again we restrict ourselves to $n = 2$ and put as before $\mathbb{Q}^2 = Q$ according to (1.230). Let $I = (0,1)$ be the unit interval on the real line $\mathbb{R}$. Let $C(I)$ and $C(Q)$ be the respective spaces of complex-valued continuous functions on $\bar{I}$ and $\bar{Q}$ according to Definition 1.24 (iii). The very classical Sobolev spaces $W^1_p(I)$, $1 < p < \infty$, can be equivalently normed by

$$\begin{aligned} \| f \,| W^1_p(I)\| &= \| f \,| L_p(I)\| + \| f' \,| L_p(I)\| \\ &\sim \max_{0 \le t \le 1} |f(t)| + \| f' \,| L_p(I)\| \\ &\sim |f(t_0)| + \| f' \,| L_p(I)\|, \quad 0 \le t_0 \le 1. \end{aligned} \tag{1.304}$$

In particular, one has the continuous embedding

$$W^1_p(I) \hookrightarrow C(I), \quad 1 < p < \infty. \tag{1.305}$$

These simple assertions can be extended to $S^1_p W(Q)$, essentially by integration. Let $\partial Q = \bigcup_{j=1}^4 I_j$ be the boundary of Q consisting of the four sides $I_1, \ldots, I_4$ (including the corner-points). Let $C(\partial Q)$ be the space of all continuous functions on ∂Q. Then

$$W^1_p(\partial Q) = \big\{ f \in C(\partial Q) : f | I_j \in W^1_p(I_j), \ j = 1,2,3,4 \big\}. \tag{1.306}$$

In particular functions $f \in W^1_p(\partial Q)$ may assumed to be continuous also in the four corner points of Q. As usual,

$$\mathrm{tr}_{\partial Q} \colon f \mapsto f|\partial Q \tag{1.307}$$

is called the *trace operator* of spaces on $\mathbb{R}^2$ or on Q into spaces on ∂Q (if exists). In our case it is well known that there is a constant $c > 0$ such that

$$\|\mathrm{tr}_{\partial Q} f \,|W^1_p(\partial Q)\| \le c\, \|f\,|S^1_p W(Q)\| \quad \text{for all } f \in S(Q), \tag{1.308}$$

where $S(Q) = S(\mathbb{R}^2)|Q$. Since $S(Q)$ is dense in $S^1_p W(Q)$, one can extend (1.308) by continuity to all functions $f \in S^1_p W(Q)$. The outcome is denoted as the corresponding trace operator. It follows that $\mathrm{tr}_{\partial Q}$ is a linear and bounded map both of $S^1_p W(\mathbb{R}^2)$ and of $S^1_p W(Q)$ *onto* $W^1_p(\partial Q)$. This can be obtained from [ST87, Theorem 2.4.2, p. 133] where one finds more information about traces of general spaces with dominating mixed smoothness. In our context it seems to be reasonable to give a direct simple proof of (1.308). By Theorem 1.67 (ii),

$$\begin{aligned}\|f\,|S^1_p W(Q)\|^* = \|f\,|L_p(Q)\| + \left\|\frac{\partial f}{\partial x_1}\,|L_p(Q)\right\| + \left\|\frac{\partial f}{\partial x_2}\,|L_p(Q)\right\| \\ + \left\|\frac{\partial^2 f}{\partial x_1 \partial x_2}\,|L_p(Q)\right\|.\end{aligned} \tag{1.309}$$

is an equivalent norm in $S^1_p W(Q)$.

Proposition 1.68. *Let $S^1_p W(Q)$ with $1 < p < \infty$ be the above Sobolev spaces with dominating mixed smoothness. Then*

$$\mathrm{id} \colon S^1_p W(Q) \hookrightarrow C(Q) \tag{1.310}$$

and

$$\mathrm{tr}_{\partial Q} \colon S^1_p W(Q) \hookrightarrow W^1_p(\partial Q). \tag{1.311}$$

Furthermore,

$$\|f\,|S^1_p W(Q)\| \sim \left(\int_Q \left|\frac{\partial^2 f(x)}{\partial x_1 \partial x_2}\right|^p \mathrm{d}x\right)^{1/p} + \|\mathrm{tr}_{\partial Q} f\,|W^1_p(\partial Q)\| \tag{1.312}$$

(*equivalent norms*).

Proof. Since $S^1_p W(Q)$ is the restriction of $S^1_p W(\mathbb{R}^2)$ to Q it follows that the restriction of $S(\mathbb{R}^2)$ to Q is dense in $S^1_p W(Q)$. Hence we can deal with smooth functions. The rest is a matter of completion. By (1.304) one obtains for fixed x_1 with $0 \le x_1 \le 1$ that

$$\begin{aligned}&|f(x_1,0)|^p + \left|\frac{\partial f}{\partial x_1}(x_1,0)\right|^p \\ &\quad\le c \int_0^1 \Big(|f(x_1,x_2)|^p + \left|\frac{\partial f}{\partial x_1}(x_1,x_2)\right|^p \\ &\qquad + \left|\frac{\partial f}{\partial x_2}(x_1,x_2)\right|^p + \left|\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1,x_2)\right|^p\Big)\mathrm{d}x_2.\end{aligned} \tag{1.313}$$

Similarly for the other three sides of Q. One can replace 0 on the left-hand side of (1.313) by x_2^0 with $0 \le x_2^0 \le 1$. Using (1.304) one has for any fixed $(x_1^0, x_2^0) \in \bar{Q}$ that

$$|f(x_1^0, x_2^0)| \le c\, \|f(\cdot, x_2^0)\, |W_p^1(I)\| \le c' \|f\, |S_p^1 W(Q)\|. \tag{1.314}$$

From (1.314) follows (1.310) and afterwards (1.311) by integration of (1.313) over x_1. By similar arguments one obtains (1.312) from

$$\int_0^1 \left| \frac{\partial f}{\partial x_1}(x_1, x_2) \right|^p \mathrm{d}x_2 \le c \left| \frac{\partial f}{\partial x_1}(x_1, 0) \right|^p + c \int_0^1 \left| \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2) \right|^p \mathrm{d}x_2 \tag{1.315}$$

and

$$\begin{aligned} \int_0^1 |f(x_1^0, x_2)|^p \,\mathrm{d}x_2 &\le c\, |f(x_1^0, 0)|^p + c \int_0^1 \left| \frac{\partial f}{\partial x_2}(x_1^0, x_2) \right|^p \mathrm{d}x_2 \\ &\le c' \,|f(x_1^0, 0)|^p + c' \int_0^1 \left| \frac{\partial f}{\partial x_2}(0, x_2) \right|^p \mathrm{d}x_2 + c' \int_Q \left| \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2) \right|^p \mathrm{d}x_1\, \mathrm{d}x_2. \end{aligned} \tag{1.316}$$

□

Remark 1.69. In connection with Faber bases in some Besov spaces and Sobolev spaces with dominating mixed smoothness we obtain later on assertions of type (1.310)–(1.312) on a larger scale based on different arguments. We are interested in the above proposition mainly for two reasons. On the one hand, the spaces $W_p^1(\mathbb{Q}^n)$ will play a dominant role in connection with two major topics of this book, numerical integration and discrepancy. On the other hand, we wanted to demonstrate how closely some assertions for spaces, say, on intervals, are related to corresponding spaces with dominating mixed smoothness in higher dimensions, say, on $\mathbb{Q}^n$.

We introduced in Definition 1.56 the spaces $S^r_{pq}A(\Omega)$ by restriction of $S^r_{pq}A(\mathbb{R}^n)$ to Ω and explained afterwards what is meant by the extension problem. So far we have satisfactory assertions for the spaces $S^r_{pp}B(\mathbb{Q}^2)$ and $S^r_p W(\mathbb{Q}^2)$ covered by Theorem 1.67. This is mainly based on the classical Hestenes extension method for spaces on intervals into corresponding spaces on $\mathbb{R}$ and the explicit norms in (1.294), (1.295) and (1.297). But this type of argument does not work for other spaces. In [T08, Chapter 4] we studied in detail the existence of (common) extension operators for isotropic spaces in rough domains. As indicated in Section 1.2.6 it might be worth to check to which extent these techniques can be used in the context of spaces with dominating mixed smoothness. This will not be done here, but as a first step in this direction we have a closer look at the spaces covered by Theorem 1.66 and their wavelet expansions. These representations can be used to construct extension operators for the spaces in both parts of Theorem 1.66. But the spaces $S^r_{pp}B(Q)$ in part (ii) of this theorem with (1.268), in particular $1 \le p < \infty$, $r > \frac{1}{p}$, are covered by Theorem 1.67 (ii) for which we have already (1.298). This justifies that we deal only with the spaces in part (i) of Theorem 1.66.

Theorem 1.70. *Let $Q = \mathbb{Q}^2$ be the unit square in $\mathbb{R}^2$ according to (1.230) with $n = 2$. Let $u \in \mathbb{N}$. Then there exists a common extension operator* ext^u*,*

$$\mathrm{ext}^u : S^r_{pp}B(Q) \hookrightarrow S^r_{pp}B(\mathbb{R}^2), \tag{1.317}$$

for all spaces $S^r_{pp}B(Q)$ with

$$\begin{cases} 0 < p \le \infty, & \max\left(\frac{1}{p}, 1\right) - 1 < r < \frac{1}{p}, \\ 1 < p < \infty, & r = 0, \\ 0 < p \le \infty, & r < 0, \end{cases} \tag{1.318}$$

and

$$u > \max\left(r, \max\left(\tfrac{1}{p}, 1\right) - 1 - r\right). \tag{1.319}$$

Proof. The conditions (1.318), (1.319) coincide with (1.263), (1.264). By Theorem 1.66 we have for these spaces the common orthogonal interior wavelet basis Ψ^Q and the expansion

$$f = \sum_{k \in \mathbb{N}_0^2} \sum_{m \in \mathbb{P}_k^\Psi} \lambda_{km} \, 2^{-(k_1+k_2)(r-\frac{1}{p})} \, \psi_{km}, \quad \lambda \in s_{pp}b(\mathbb{Z}^Q), \tag{1.320}$$

with $\lambda_{km} = \lambda_{km}(f)$ as in (1.277). Recall that

$$\psi_{km}(x) = \psi_{k_1m_1}(x_1)\, \psi_{k_2m_2}(x_2), \quad k \in \mathbb{N}_0^2, \; m_l = 1, \ldots, N_{k_l}, \tag{1.321}$$

where $\psi_{jl}(t)$ are the wavelets from Definition 1.60 and Theorem 1.62. For the interval $I = \Omega$ one can identify the wavelets ψ_{jl} with Φ^j_l used in [T08, Theorems 3.13, 3.23, pp. 80/81, 89/90]. These wavelets originate from [T08, Definition 2.4, p. 32] having either the desired moment conditions (interior wavelets) or can be complemented such that these modified wavelets have the desired moment conditions (boundary wavelets) as described in [T08, p. 81]. This can be used to construct extension operators for $B^r_{pp}(I)$ to $B^r_{pp}(\mathbb{R})$, [T08, p. 105]. By (1.321) one has now the needed moment conditions for ψ_{km} as required in Definition 1.42, (1.175). Then one can construct an extension operator for the spaces $S^r_{pp}B(Q)$ in the same way as for $B^r_{pp}(I)$. □

1.3 Logarithmic spaces

1.3.1 Introduction

A compact set Γ in $\mathbb{R}^n$ is called a d-set, $0 < d < n$, if there is a Radon measure μ in $\mathbb{R}^n$ with

$$\operatorname{supp} \mu = \Gamma \quad \text{and} \quad \mu\big(B(\gamma, t)\big) \sim t^d \tag{1.322}$$

for all balls $B(\gamma, t)$ in $\mathbb{R}^n$ centred at $\gamma \in \Gamma$ and of radius t, $0 < t < 1$. This is a distinguished class of fractal sets in $\mathbb{R}^n$ which has been studied especially in the last two decades with great intensity. The standard references are [Fal85], [Mat95]. We dealt with these fractal sets in [T97], [T01], [T06] in the framework of function spaces. Especially the isotropic Besov spaces $B^s_{pq}(\mathbb{R}^n)$ fit perfectly to d-sets. This is also illustrated by some dichotomy assertions which may be found in [T08, Section 6.4]. In connection with fractal drums (*the music of the ferns*) it came out that perturbed compact d-sets Γ in $\mathbb{R}^n$ are of interest for which there is a Radon measure μ in $\mathbb{R}^n$ with

$$\operatorname{supp} \mu = \Gamma, \quad \mu\big(B(\gamma, t)\big) \sim t^d \, \Psi(t), \tag{1.323}$$

$\gamma \in \Gamma$, $0 < t < 1$. Here again $0 < d < n$, whereas $\Psi(t)$ is a monotone (decreasing or increasing) positive function for $0 \le t \le 1$ with $\Psi(t) \sim \Psi(t^2)$. The most distinguished cases are $\Psi_b(t) = (1 + |\log t|)^b$, $b \in \mathbb{R}$, where log is always taken to base 2. The search for optimally adapted spaces results in spaces $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ of generalised smoothness of the same type as in Definition 1.1 but with

$$\|f \,|B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|_\varphi = \Big(\sum_{j=0}^{\infty} 2^{jsq} \Psi(2^{-j})^q \, \|(\varphi_j \hat{f}\,)^\vee \,|L_p(\mathbb{R}^n)\|^q \Big)^{1/q} \tag{1.324}$$

in place of (1.9). If $\Psi = \Psi_b$ then $\Psi_b(2^{-j}) \sim (1 + j)^b$. This may explain the notation *logarithmic spaces* if the perturbation of 2^{js} in (1.324) is of power type, $\sim (1+j)^b$. The theory of these spaces started in [EdT98], [EdT99], but the first comprehensive studies are due to S. D. Moura, [Mou01], and (in the more general context of so-called h-sets) to M. Bricchi, [Bri04] (and his preceding papers mentioned there). We do not deal with fractals in this book, but the situation is similar now as some ten years ago when asking for optimally adapted spaces for perturbed d-sets. The main topic of this book is sampling, numerical integration and discrepancy (based on Haar and Faber bases). Spaces with dominating mixed smoothness (in cubes and in more general domains) are an adequate and effective but maybe not optimal choice. Taking assertions in one dimension, for intervals, as a guide one has in higher dimensions additional log-factors. One may try to compensate these log-terms by switching from spaces with dominating mixed smoothness to their logarithmic perturbations imitating the replacement of (1.9) by (1.324). First we collect in Section 1.3.2 some results for logarithmic spaces in one dimension preferably with $\Psi = \Psi_b$ in the above notation. This will be combined in Section 1.3.4 with some assertions obtained in Section 1.2 for spaces with dominating mixed smoothness. We will be a little bit sketchy (as in [EdT98], [EdT99] some ten years ago) hoping that everything (and more) will be substantiated later on (as in connection with [EdT98], [EdT99]). Just for this purpose we sandwich Section 1.3.3 between Sections 1.3.2 and 1.3.4 which might be considered as a research proposal.

Remark 1.71. As indicated above we restrict ourselves to the bare minimum of a few assertions which will be helpful for the later considerations. On the other hand the recent Fourier-analytical theory of (isotropic) spaces of generalised smoothness produced in the last decade many deep and interesting assertions about atoms, wavelets, (sharp) embeddings, envelopes and related topics. In addition to the above-mentioned papers [Mou01], [Bri04] we refer to [BrM03], [CaF06], [CaH05], [CaL06], [CaL09], [CaM04a], [CaM04b], [FaL06], [HaM04], [HaM08], [KnZ06], [Lop09], [Mou07], [MNP09] and the relevant parts in [Har07] based on the just-mentioned papers. It might be of interest not only for its own sake but also for the applications we have in mind and which are subject of this book to develop a corresponding theory for logarithmic spaces with dominating mixed smoothness.

1.3.2 Logarithmic spaces on $\mathbb{R}$

By (1.324) it is quite clear what is meant by isotropic logarithmic spaces on $\mathbb{R}^n$. But we restrict ourselves in this section to $n = 1$, hence logarithmic spaces on the real line $\mathbb{R}$. This is the only case of interest for us later on when it comes to logarithmic spaces with dominating mixed smoothness in $\mathbb{R}^n$ and on cubes $\mathbb{Q}^n$ (with a preference of $n = 2$). First we formalise what has been said above.

Definition 1.72. Let $\varphi = \{\varphi_j\}_{j=0}^{\infty}$ be the dyadic resolution of unity on $\mathbb{R}$ according to (1.5)–(1.7) (or (1.146), (1.147)). Let $0 < p \leq \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, and $b \in \mathbb{R}$. Then $B^{s,b}_{pq}(\mathbb{R})$ is the collection of all $f \in S'(\mathbb{R})$ such that

$$\|f\,|B^{s,b}_{pq}(\mathbb{R})\|_{\varphi} = \Big(\sum_{j=0}^{\infty} 2^{jsq}(1+j)^{bq}\|(\varphi_j \hat{f})^{\vee}|L_p(\mathbb{R})\|^q\Big)^{1/q} < \infty \tag{1.325}$$

(with the usual modification if $q = \infty$).

Remark 1.73. If $b = 0$ then the above definition and its obvious n-dimensional extension coincide with Definition 1.1 (i) There is an F-counterpart mentioned later on in an n-dimensional setting. These (and more general) spaces have been studied in detail in [Mou01], including atomic and subatomic characterisations, local means, liftings and embeddings. As for the further developments and some generalisations we refer to the papers mentioned in the preceding Section 1.3.1. These spaces are independent of φ (equivalent quasi-norms). We collect a few specific properties which will be of some use for us later on and which are (more or less) covered by the literature. Let

$$(\Delta^1_h f)(x) = f(x+h) - f(x), \quad (\Delta^{l+1}_h f)(x) = \Delta^1_h(\Delta^l_h f)(x), \tag{1.326}$$

where $x \in \mathbb{R}$, $h \in \mathbb{R}$, be the same differences as used before.

Proposition 1.74. *Let* $0 < p, q \leq \infty$, $b \in \mathbb{R}$, *and*

$$\max\Big(\frac{1}{p}, 1\Big) - 1 < s < M \in \mathbb{N}. \tag{1.327}$$

Then

$$\begin{aligned}
&\|f\,|B^{s,b}_{pq}(\mathbb{R})\| \\
&\sim \|f\,|L_p(\mathbb{R})\| + \Big(\int_0^1 t^{-sq}(1+|\log t|)^{bq} \sup_{0<h<t} \|\Delta_h^M f\,|L_p(\mathbb{R})\|^q \frac{dt}{t}\Big)^{1/q} \\
&\sim \|f\,|L_p(\mathbb{R})\| + \Big(\int_0^1 h^{-sq}(1+|\log h|)^{bq} \|\Delta_h^M f\,|L_p(\mathbb{R})\|^q \frac{dh}{h}\Big)^{1/q}
\end{aligned} \tag{1.328}$$

are equivalent quasi-norms in $B^{s,b}_{pq}(\mathbb{R})$*. If, in addition,* $s > 1/p$*, then also*

$$\begin{aligned}
&\|f\,|B^{s,b}_{pq}(\mathbb{R})\| \\
&\sim \|f\,|L_p(\mathbb{R})\| + \Big(\int_0^1 t^{-sq}(1+|\log t|)^{bq} \big\| \sup_{0<h<t} |\Delta_h^M f(\cdot)|\,|L_p(\mathbb{R})\|^q \frac{dt}{t}\Big)^{1/q} \\
&\sim \|f\,|L_p(\mathbb{R})\| + \Big(\int_0^1 t^{-sq}(1+|\log t|)^{bq} \Big\| \sup_{\substack{0<h<t \\ |y-\cdot|<t}} |(\Delta_h^M f)(y)|\,|L_p(\mathbb{R})\Big\|^q \frac{dt}{t}\Big)^{1/q}
\end{aligned} \tag{1.329}$$

are equivalent quasi-norms in $B^{s,b}_{pq}(\mathbb{R})$.

Remark 1.75. If $b = 0$, hence $B^{s,0}_{pq}(\mathbb{R}) = B^s_{pq}(\mathbb{R})$, then (1.328) is covered by (1.23), (1.24) and [T83, Theorem 2.5.12, p. 110]. Otherwise the first equivalence in (1.328) is a special case of [Mou07, Theorem 4.1, p. 1196] dealing with characterisations of this type for larger classes of spaces with generalised smoothness. We refer also to [HaM04, Theorem 2.5, Example 2.6, pp. 161/162], [HaM08], [FaL06, Section 3.3, pp. 27/28], the references given there to the extensive Russian literature, and most recently and most generally to [CaL09, Theorem 3.16]. From this assertion one can prove the second equivalence in (1.328) in the same way as in [T83, Section 2.5.12, p. 112] where we dealt with the case $b = 0$. This has been done in detail in [CaL09, Theorem 3.16] for more general Besov spaces of the above type. Recall that

$$B^{s,b}_{pq}(\mathbb{R}) \hookrightarrow C(\mathbb{R}) \quad \text{if } s > 1/p \tag{1.330}$$

as in case of $b = 0$ and with the same proof. Then (1.329) makes sense. The first equivalence in (1.329) with $b = 0$ is covered by [T92, Theorem 3.5.3, p. 194] (with $u = \infty$ there). This can be extended to $b \in \mathbb{R}$ (all what one needs is available). But the proof of this theorem covers also the second equivalence in (1.329).

Remark 1.76. Let again $I = (0,1)$ be the unit interval in $\mathbb{R}$ and let $B^{s,b}_{pq}(I)$ be the restriction of $B^{s,b}_{pq}(\mathbb{R})$ to I as in Definition 1.24. As in Section 1.1.6 one may ask for linear and bounded extension operators and intrinsic characterisations. Let again

$$0 < p, q \le \infty, \quad \max\big(\tfrac{1}{p}, 1\big) - 1 < s < M \in \mathbb{N}, \ b \in \mathbb{R}. \tag{1.331}$$

The classical Hestenes extension method mentioned in Step 2 of the proof of Theorem 1.67 with a reference to [T92, Section 4.5.2] works also for the corresponding spaces $B^{s,b}_{pq}(I)$. This follows from interpolation with function parameters. We refer to Point 8 of the subsequent Section 1.3.3 dealing with extensions in an n-dimensional setting. Then there should be also counterparts of the equivalent quasi-norms (1.289), (1.290). Hence one can expect by (1.328) that

$$\begin{aligned}
&\|f\,|B^{s,b}_{pq}(I)\| \\
&\quad\sim \|f\,|L_p(I)\| + \Big(\int_0^1 t^{-sq}(1+|\log t|)^{bq} \sup_{0<h<t} \|\Delta^M_{h,I} f\,|L_p(I)\|^q \,\frac{\mathrm{d}t}{t}\Big)^{1/q} \\
&\quad\sim \|f\,|L_p(I)\| + \Big(\int_0^1 h^{-sq}(1+|\log h|)^{bq} \|\Delta^M_{h,I} f\,|L_p(I)\|^q \,\frac{\mathrm{d}h}{h}\Big)^{1/q}
\end{aligned} \tag{1.332}$$

are equivalent quasi-norms in $B^{s,b}_{pq}(I)$. We will not need these assertions later on when dealing with Haar bases and Faber bases for spaces of this type. One may consider the above remarks as part of the research proposal of the following section. The direct arguments in [HaT08, Sections 3.4, 4.2] may be helpful in this context at least for some spaces.

1.3.3 Isotropic logarithmic spaces: comments, problems, proposals

Logarithmic spaces with dominating mixed smoothness in $\mathbb{R}^n$, $n \ge 2$, (preferably $\mathbb{R}^2$) are based on logarithmic spaces on the real line $\mathbb{R}$ as considered in the preceding Section 1.3.2. However our original motivation to deal with spaces of generalised smoothness is truly n-dimensional and closely related to traces of these spaces on compact isotropic fractal sets, d-sets and (d,Ψ)-sets according to (1.322) (1.323), and more generally, h-sets. This goes back to [EdT98], [EdT99] and is also well reflected by the two basic papers [Mou01], [Bri04] in this context. In the subsequent papers as listed at the end of Section 1.3.1 the connection with fractal geometry and related spectral theory for fractal elliptic operators was no longer in the focus of interest. These papers deal with spaces of generalised smoothness in $\mathbb{R}^n$ for their own sake, concentrating on atomic and subatomic decompositions, local means, (sharp) embeddings and (above all) impressive deep assertions about growth and continuity envelopes. We restricted the list at the end of Section 1.3.1 to papers directly related to the indicated topics. But there is a huge literature about spaces with generalised smoothness, quite often defined in terms of differences $\Delta^M_h f$ of functions and moduli of continuity, especially by Russian and Czech mathematicians and their co-workers. We collected respective references in [T06, Section 1.9.5, pp. 52–55] including the surveys [KaL87], [Liz86] reflecting the extensive Russian literature up to the end of the 1980s. A more recent comprehensive list may also be found in [FaL06] which has the character of a survey also with respect to the quoted literature. This will not be repeated here. As far as we can see all these papers deal with spaces on $\mathbb{R}^n$. There is no attempt to extend the theory of the spaces $A^s_{pq}(\Omega)$ on domains Ω in $\mathbb{R}^n$ to corresponding isotropic spaces of generalised smoothness. To avoid a misunderstanding we add a comment. Let Γ

be a compact (d,Ψ)-set in $\mathbb{R}^n$ according to (1.323). Then it was just one of the main reasons of [EdT98], [EdT99] and also of some subsequent papers, especially [Mou01], [Bri04] to introduce logarithmic spaces on Γ by traces of suitable spaces on $\mathbb{R}^n$,

$$B^{(s,\Psi^a)}_{pq}(\Gamma) = \mathrm{tr}_\Gamma\, B^{(s+\frac{n-d}{p},\Psi^{\frac{1}{p}+a})}_{pq}(\mathbb{R}^n), \tag{1.333}$$

where

$$0 < p,q \le \infty, \quad s > 0, \quad a \in \mathbb{R}. \tag{1.334}$$

Here tr_Γ must be understood similarly as in (1.307), (1.308). We refer to [T01, Section 22.20, p. 346] and more general to [T01, Sections 22, 23] for the fractal connection of these spaces and the music of corresponding isotropic drums. This may also be of some interest in our context but it is not what we meant by the above remarks about spaces of type $A^{(s,\Psi)}_{pq}(\Omega)$ where Ω is a domain in $\mathbb{R}^n$. What follows might be considered as a *research proposal*. We restrict ourselves to keywords and a few comments. But first we formalise some definitions.

A real function Ψ on the interval $(0,1]$ is called *admissible* if it is positive and monotone on $(0,1]$ and if

$$\Psi(2^{-j}) \sim \Psi(2^{-2j}), \quad j \in \mathbb{N}_0. \tag{1.335}$$

Let $0 < c < 1$ and $b \in \mathbb{R}$. Then

$$\Psi(t) = \Psi_b(t) = |\log ct|^b, \quad 0 < t \le 1, \tag{1.336}$$

are distinguished examples. One has in any case

$$\Psi(t) \sim \Psi(t^2), \quad 0 < t \le 1. \tag{1.337}$$

Furthermore, there are positive numbers c_1, c_2, b and c with $0 < c < 1$ such that

$$c_1\, |\log ct|^{-b} \le \Psi(t) \le c_2\, |\log ct|^b, \quad 0 < t \le 1. \tag{1.338}$$

We refer to [T01, pp. 333/334] or, in greater detail, to [Mou01] and the above-mentioned subsequent papers. This may explain where the notation *logarithmic spaces* comes from.

Definition 1.77. Let Ψ be an admissible function.

(i) Let $\varphi = \{\varphi_j\}_{j=0}^\infty$ be the dyadic resolution of unity in $\mathbb{R}^n$ according to (1.5)–(1.7). Let

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}, \tag{1.339}$$

(with $p < \infty$ for the F-spaces). Then $B^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \,|B^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|_\varphi = \Big(\sum_{j=0}^\infty 2^{jsq}\Psi(2^{-j})^q \|(\varphi_j \hat{f})^\vee |L_p(\mathbb{R}^n)\|^q\Big)^{1/q} < \infty \tag{1.340}$$

and $F^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$\|f \,|F^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|_\varphi = \Big\| \Big(\sum_{j=0}^{\infty} 2^{jsq}\, \Psi(2^{-j})^q \, |(\varphi_j \hat{f})^\vee(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\| < \infty \tag{1.341}$$

(with the usual modification if $q = \infty$).

(ii) Let Ω be a domain (= open set) in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$, let p, q, s be as in (1.339) (with $p < \infty$ for the F-spaces) and let either $A = B$ or $A = F$. Then

$$A^{(s,\Psi)}_{pq}(\Omega) = \big\{ f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in A^{(s,\Psi)}_{pq}(\mathbb{R}^n) \big\}, \tag{1.342}$$

$$\|f \,|A^{(s,\Psi)}_{pq}(\Omega)\| = \inf \|g \,|A^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|, \tag{1.343}$$

where the infimum is taken over all $g \in A^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ with $g|\Omega = f$. Let

$$\widetilde{A}^{(s,\Psi)}_{pq}(\overline{\Omega}) = \big\{ f \in A^{(s,\Psi)}_{pq}(\mathbb{R}^n) : \operatorname{supp} f \subset \overline{\Omega} \big\}. \tag{1.344}$$

Then

$$\widetilde{A}^{(s,\Psi)}_{pq}(\Omega) = \big\{ f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in \widetilde{A}^{(s,\Psi)}_{pq}(\overline{\Omega}) \big\}, \tag{1.345}$$

$$\|f \,|\widetilde{A}^{(s,\Psi)}_{pq}(\Omega)\| = \inf \|g \,|A^{(s,\Psi)}_{pq}(\mathbb{R}^n)\|, \tag{1.346}$$

where the infimum is taken over all $g \in \widetilde{A}^{(s,\Psi)}_{pq}(\overline{\Omega})$ with $g|\Omega = f$.

Remark 1.78. This is the generalisation of Definitions 1.1, 1.24. We used the same notation as there. We refer to the Remarks 1.2, 1.25 for comments. If $\Psi = \Psi_b$ as in (1.336) then it is usual to write

$$A^{s,b}_{pq}(\mathbb{R}^n) = A^{(s,\Psi_b)}_{pq}(\mathbb{R}^n) \tag{1.347}$$

used now also with Ω in place of $\mathbb{R}^n$. On the other hand, some generalisations of the above spaces $A^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ have been considered, originally in connection with fractal h-sets introduced by M. Bricchi, [Bri02], [Bri04], later on in [FaL06] and also in some of the papers mentioned at the end of Section 1.3.1. One replaces $2^{js}\Psi(2^{-j})$ in (1.340), (1.341) by σ_j and 2^k in the resolution of unity (1.6) by some N_k, typically subject of compatibility conditions of type $\sigma_j \sim \sigma_{j+1}$ and $N_k \sim N_{k+1}$. However to which extent the comments and suggestions below for the spaces $A^{(s,\Psi)}_{pq}$ apply also to these more general spaces remains to be seen.

We add now a few further comments about the above spaces $A^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ and $A^{(s,\Psi)}_{pq}(\Omega)$ restricting ourselves to keywords.

1. Atoms. The atomic expansion Theorem 1.7 for the spaces $A^s_{pq}(\mathbb{R}^n)$ can surely be extended to the spaces $A^{(s,\Psi)}_{pq}(\mathbb{R}^n)$. This is essentially covered by [Mou01, Theorem 1.18, p. 31], including the minor technical modifications in Definition 1.5 which are useful in connection with Haar bases (a_{jm} are L_∞-functions) and Faber bases (Lipschitz condition (1.32)).

2. Local means. Characterisations of the spaces $A^s_{pq}(\mathbb{R}^n)$ in terms of local means go back to [T92, pp. 58, 122, 138] and the original paper [Tri88]. This version has been extended to the spaces $A^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ in [Mou01, Theorem 1.10]. But in connection with spaces $A^s_{pq}(\Omega)$ in rough domains, the extension problem and wavelet characterisations as considered in [T08] we needed the much stronger Theorem 1.15 with a reference to [T08, Theorem 1.15, p. 7]. It would be desirable to find a counterpart of this assertion for the spaces $A^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ (maybe based on the proof in [T08, pp. 8–12]).

3. Wavelets. The wavelet characterisation of the spaces $A^s_{pq}(\mathbb{R}^n)$ according to Theorem 1.18 is mainly based on sharp assertions about atoms and local means. In other words, if one has sufficiently strong counterparts of these assertions for the spaces $A^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ then one has a good chance to extend Theorem 1.18 to the spaces $A^{(s,\Psi)}_{pq}(\mathbb{R}^n)$. As for an alternative proof for Besov spaces based on interpolation we refer to [Alm05].

4. Diffeomorphisms, pointwise multipliers. The step from $A^s_{pq}(\mathbb{R}^n)$ to $A^s_{pq}(\Omega)$, where Ω might be a bounded smooth domain, requires that the spaces $A^s_{pq}(\mathbb{R}^n)$ can be localised, that smooth cut-off functions are pointwise multipliers, and that they can locally distorted by smooth diffeomorphisms. This has been finally established for the spaces $A^s_{pq}(\mathbb{R}^n)$ in [T92, Sections 4.2, 4.3, pp. 201–211]. One may asked for an extension of these assertions to the spaces $A^{(s,\Psi)}_{pq}(\mathbb{R}^n)$.

5. Traces. Similarly as for the spaces $A^s_{pq}(\mathbb{R}^n)$ one asks for traces of $f \in A^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ and, more generally, of $\frac{\partial^l f}{\partial x_n^l}$ on the hyper-plane $\mathbb{R}^{n-1}$,

$$\mathrm{tr}^l : \frac{\partial^l f}{\partial x_n^l}(x) \mapsto \frac{\partial^l f}{\partial x_n^l}(x',0), \quad x = (x', x_n),\ x' \in \mathbb{R}^{n-1}. \tag{1.348}$$

Traces have a long history. One may consult [T78], [T83], [T92]. But nowadays one can base trace problems on atoms and wavelets. The construction of wavelet-friendly extension operators as in [T08, Section 5.1.3] seems to be especially effective.

6. Spaces on manifolds. Having affirmative answers of the above questions for the spaces $A^{(s,\Psi)}_{pq}(\mathbb{R}^n)$ then one can construct by standard techniques spaces $A^{(s,\Psi)}_{pq}(\Gamma)$ on compact n-dimensional C^∞ manifolds Γ, including wavelet frames (and in some cases even wavelet bases). As for the spaces $A^{\varrho}_{pq}(\Gamma)$ we refer to [T08, Section 5.1.2].

7. Homogeneity, extension. Let

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s > n\Big(\max\Big(\frac{1}{p}, \frac{1}{q}, 1\Big) - 1\Big). \tag{1.349}$$

Then

$$\|f(\lambda\cdot)\,|F^s_{pq}(\mathbb{R}^n)\| \sim \lambda^{s-\frac{n}{p}}\,\|f\,|F^s_{pq}(\mathbb{R}^n)\|, \quad \mathrm{supp}\, f \subset \{x \in \mathbb{R}^n : |x| \le \lambda\} \tag{1.350}$$

where the equivalence constants are independent of f and λ with $0 < \lambda \leq 1$. This goes back to [T01, Corollary 5.16, p. 66]. There are counterparts for the B-spaces, [CLT07], and more general results in [T08, Theorem 2.11, p. 34]. This homogeneity assertion plays a decisive role in connection with wavelet representations for the spaces $\widetilde{A}^s_{pq}(\Omega)$ and $A^s_{pq}(\Omega)$ according to Definition 1.24 and the crucial extension problem. Unfortunately there is no immediate counterpart for the spaces $A^{(s,\Psi)}_{pq}(\Omega)$ and $\widetilde{A}^{(s,\Psi)}_{pq}(\Omega)$ as introduced in Definition 1.77. But according to [CaL09, Theorem 3.18] there is a substitute at least for some B-spaces. It remains to be seen whether the theory developed in [T08] about spaces of type $A^s_{pq}(\Omega)$ in rough domains, wavelet expansions and extensions from Ω to $\mathbb{R}^n$ has a counterpart for corresponding spaces $A^{(s,\Psi)}_{pq}(\Omega)$.

8. Interpolation, extension. The classical real interpolation method has been extended to interpolation methods with function parameters. This has some history. We refer to [Gus78], [Jan81], [Mer84], [Per86], [CoF88], [CFMM07]. Of special interest for us are [Mer84], [CoF88] extending the real interpolation method $(A_0, A_1)_{\theta,q}$ with $0 < \theta < 1$ to interpolation methods $(A_0, A_1)_{g(\cdot),q}$ where $g(\cdot)$ is a function parameter covering in particular logarithmic spaces. This can be applied to the above logarithmic spaces

$$B^{(s,\Psi)}_{pq}(\mathbb{R}^n) = \big(B^{s_0}_{pq_0}(\mathbb{R}^n),\ B^{s_1}_{pq_1}(\mathbb{R}^n)\big)_{g(\cdot),q}, \quad s_0 < s < s_1, \tag{1.351}$$

$p, q_0, q_1, q \in (0, \infty]$ and some admitted function parameter g in dependence on Ψ (and s_0, s_1, s). We refer to [CaM04a, p. 47] and [Alm05, Proposition 7, pp. 204/205] based on [Mer84, Theorem 13, p. 194] and [CoF88, Theorem 5.3, Remark 5.4]. One may ask for an Ω-counterpart of (1.351), hence

$$B^{(s,\Psi)}_{pq}(\Omega) = \big(B^{s_0}_{pq_0}(\Omega),\ B^{s_1}_{pq_1}(\Omega)\big)_{g(\cdot),q}, \tag{1.352}$$

and the existence of a (bounded linear) extension operator

$$\mathrm{ext}\colon\ B^{(s,\Psi)}_{pq}(\Omega) \hookrightarrow B^{(s,\Psi)}_{pq}(\mathbb{R}^n). \tag{1.353}$$

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, or a bounded interval on the real line. Then there is a universal extension operator,

$$\mathrm{ext}\colon\ B^s_{pq}(\Omega) \hookrightarrow B^s_{pq}(\mathbb{R}^n) \tag{1.354}$$

in the understanding of Theorem 1.27 and (1.99)–(1.102). One has in particular

$$\mathrm{re} \circ \mathrm{ext} = \mathrm{id} \quad \big(\text{identity in } B^s_{pq}(\Omega)\big). \tag{1.355}$$

Then it follows from (1.351) and the interpolation property that

one has (1.353) *and also* (1.352) *for all* $s \in \mathbb{R}$ *and* $p, q_0, q_1, q \in (0, \infty]$.

In [T08, Chapter 4] we dealt with several types of common extension operators in rough domains (beyond bounded Lipschitz domains) for the spaces $A^s_{pq}(\Omega)$. *This can be extended by the above arguments to spaces of type* $B^{(s,\Psi)}_{pq}(\Omega)$.

9. Elliptic boundary value problems. Let Ω be a bounded C^∞ domain in $\mathbb{R}^n$ and let $\Gamma = \partial\Omega$ be its boundary. Elliptic boundary value problems of type

$$Au = \sum_{|\alpha| \le 2m} a_\alpha(x)\, D^\alpha u = f \quad \text{in } \Omega, \tag{1.356}$$

$$B_j u = \sum_{|\beta| \le k_j} b_{j\beta}(\gamma)\, \mathrm{tr}_\Gamma\, D^\beta u = g_j \quad \text{on } \Gamma, \tag{1.357}$$

$j = 1, \ldots, m$ and $k_j \in \mathbb{N}_0$ with $0 \le k_1 < k_2 < \cdots < k_m < 2m$ have a long history. About conditions for A and B_j we refer to [T78], [T83] and the huge literature mentioned there. One may think about the Dirichlet problem,

$$A = (-\Delta)^m \quad \text{and} \quad B_j = \frac{\partial^j}{\partial \mu^j}, \quad m \in \mathbb{N};\ j = 0, \ldots, m-1, \tag{1.358}$$

as a prototype. Here $\Delta = \sum_{l=1}^n \frac{\partial^2}{\partial x_l^2}$ is the Laplacian, whereas μ denotes the outer normal. The L_2-theory has been developed in the famous book by S. Agmon, [Agm65]. An introduction to boundary value problems for second order elliptic equations may also be found in [HaT08]. These assertions have been extended to L_p-spaces, $1 < p < \infty$, Hölder–Zygmund spaces $\mathcal{C}^s$, $s > 0$, and finally to the spaces B^s_{pq} and F^s_{pq}. The corresponding theory in full generality is due to [FrR95] and subject of [RuS96, Chapter 3]. Furnished with the interpolation (1.352) one has a good chance to extend this theory from some B^s_{pq}-spaces to corresponding $B^{(s,\Psi)}_{pq}$-spaces. This is of interest for its own sake. But it may be also of some use in numerical applications.

10. Compact embeddings. Entropy numbers, approximation numbers and other widths of compact embeddings between spaces of type $A^{(s,\Psi)}_{pq}(\Omega)$ in bounded domains Ω are not only of interest for their own sake but they can also be used for spectral assertions (distribution of eigenvalues) of the above elliptic operators. Entropy numbers for compact embeddings between spaces $B^{s,b}_{pq}(\Omega)$ in bounded domains Ω in limiting situations have been considered in [Leo00]. Some assertions about approximation numbers for embeddings of $A^{(s,\Psi)}_{pq}(\Omega)$ into $C(\Omega)$ may be found in [CaH05, Proposition 5.10, pp. 65/66]. Interesting sharp results for compact embeddings between $B^{(s,\Psi)}_{pq_1}(\Omega)$ and $B^s_{pq_2}(\Omega)$ have been obtained recently in [CoK09]. For the approximation numbers one needs the extension property (1.353) for the spaces $B^{(s,\Psi)}_{pq}(\Omega)$. If, in addition, the embedding

$$\mathrm{id}\colon B^{(s,\Psi)}_{pq}(\Omega) \hookrightarrow C(\Omega) \tag{1.359}$$

is compact, then one may also ask for the behaviour of linear and non-linear sampling numbers.

11. Approximation of elliptic problems. In numerics one asks for optimal approximations of elliptic problems in bounded (smooth or Lipschitz) domains Ω in $\mathbb{R}^n$ by distinguished linear and non-linear mappings. This has been done in [DNS06a], [DNS06b], [DNS07], [DNS09], [Vyb07] in the context of Sobolev spaces $H^s(\Omega) = H^s_2(\Omega)$ and Besov spaces $B^s_{pq}(\Omega)$. If some of the above points have affirmative answers then one may replace $B^s_{pq}(\Omega)$ by $B^{(s,\Psi)}_{pq}(\Omega)$ hoping for a finer tuning.

There are now three motivations to deal with logarithmic Besov spaces.

- They can be created by the real interpolation (1.351), (1.352) from related Besov spaces.
- They originate naturally from fractal geometry, (1.323), (1.324).
- Combined with dominating mixed smoothness and restricted to domains, preferably cubes, they may be of some use in numerical integration and discrepancy.

Nevertheless they will play only a marginal role in this book. But this may change in future.

1.3.4 Logarithmic spaces with dominating mixed smoothness

On the one hand we introduced in Definition 1.38 the spaces $S^r_{pq}A(\mathbb{R}^n)$ with dominating mixed smoothness where preference is given to $n = 2$. According to Definition 1.56 corresponding spaces $S^r_{pq}A(\Omega)$ on domains Ω in $\mathbb{R}^n$ are defined by restriction where $\Omega = \mathbb{Q}^n$ in (1.230) is of special interest, again with a preference of $n = 2$. On the other hand we dealt in the preceding Section 1.3.3 with isotropic logarithmic spaces on $\mathbb{R}^n$ and in domains. In the context of numerical integration and discrepancy it is reasonable to admit logarithmic perturbations of some spaces with dominating mixed smoothness preferably in $\mathbb{Q}^n$. Then we rely on specific arguments based on Haar bases and Faber bases. This may explain why we restrict ourselves now to basic definitions and a few comments.

Definition 1.79. Let $\varphi = \{\varphi_k\}_{k\in\mathbb{N}^2_0}$ be the dyadic resolution of unity according to (1.146)–(1.149).

(i) Let

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad r \in \mathbb{R}, \quad b \in \mathbb{R}. \tag{1.360}$$

Then $S^{r,b}_{pq}B(\mathbb{R}^2)$ is the collection of all $f \in S'(\mathbb{R}^2)$ such that

$$\begin{aligned} &\|f\,|S^{r,b}_{pq}B(\mathbb{R}^2)\|_\varphi \\ &= \Big(\sum_{k\in\mathbb{N}^2_0} 2^{r(k_1+k_2)q}(1+k_1)^{bq}(1+k_2)^{bq}\big\|(\varphi_k\hat{f})^\vee|L_p(\mathbb{R}^2)\big\|^q\Big)^{1/q} < \infty \end{aligned} \tag{1.361}$$

(with the usual modification if $q = \infty$).

(ii) Let

$$0 < p < \infty, \quad 0 < q \le \infty, \quad r \in \mathbb{R}, \quad b \in \mathbb{R}. \tag{1.362}$$

Then $S^{r,b}_{pq}F(\mathbb{R}^2)$ is the collection of all $f \in S'(\mathbb{R}^2)$ such that

$$\begin{aligned} &\|f \,|S^{r,b}_{pq}F(\mathbb{R}^2)\|_\varphi \\ &\quad = \Big\| \Big(\sum_{k\in\mathbb{N}_0^2} 2^{r(k_1+k_2)q}(1+k_1)^{bq}(1+k_2)^{bq}|(\varphi_k \hat{f})^\vee(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^2)\Big\| \\ &\quad < \infty \end{aligned} \tag{1.363}$$

(with the usual modification if $q = \infty$).

Remark 1.80. These logarithmic spaces with dominating mixed smoothness generalise Definition 1.38. We refer also to Definitions 1.72, 1.77 (i). These quasi-Banach spaces are independent of φ (equivalent quasi-norms). Definition 1.77 (ii) can be modified as follows.

Definition 1.81. Let Ω be a domain (= open set) in $\mathbb{R}^2$ with $\Omega \neq \mathbb{R}^2$. Let p, q, r, b be as in (1.360) (with $p < \infty$ for the F-spaces) and let either $A = B$ or $A = F$. Then

$$S^{r,b}_{pq}A(\Omega) = \big\{ f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in S^{r,b}_{pq}A(\mathbb{R}^2) \big\}, \tag{1.364}$$

$$\|f \,|S^{r,b}_{pq}A(\Omega)\| = \inf \|g \,|S^{r,b}_{pq}A(\mathbb{R}^2)\|, \tag{1.365}$$

where the infimum is taken over all $g \in S^{r,b}_{pq}A(\mathbb{R}^2)$ with $g|\Omega = f$. Let

$$\widetilde{S}^{r,b}_{pq}A(\overline{\Omega}) = \big\{ f \in S^{r,b}_{pq}A(\mathbb{R}^2) : \operatorname{supp} f \subset \overline{\Omega} \big\}. \tag{1.366}$$

Then

$$\widetilde{S}^{r,b}_{pq}A(\Omega) = \big\{ f \in D'(\Omega) : f = g|\Omega \text{ for some } g \in \widetilde{S}^{r,b}_{pq}A(\overline{\Omega}) \big\}, \tag{1.367}$$

$$\|f \,|\widetilde{S}^{r,b}_{pq}A(\Omega)\| = \inf \|g \,|S^{r,b}_{pq}A(\mathbb{R}^2)\| \tag{1.368}$$

where the infimum is taken over all $g \in \widetilde{S}^{r,b}_{pq}A(\overline{\Omega})$ with $g|\Omega = f$.

Remark 1.82. Similarly as in Section 1.2.5 there are obvious extensions of the above definitions to higher dimensions. If $b = 0$ then $S^{r,0}_{pq}A(\Omega) = S^r_{pq}A(\Omega)$ are the spaces introduced in Definition 1.56. As in (1.227), (1.228) one may ask for the existence of linear extension operators. But the discussions in Section 1.2.6 apply even more to the spaces $S^{r,b}_{pq}A(\Omega)$ suggesting to deal preferably with $\Omega = \mathbb{Q}^n$ according to (1.230) (or more specifically $\Omega = \mathbb{Q}^2$). It remains to be seen to which extent the assertions from Sections 1.2.7, 1.2.8 can be extended to the above spaces, especially to $S^{r,b}_{pp}B(\mathbb{R}^2)$ and $S^{r,b}_{pp}B(\mathbb{Q}^2)$.

Later on we deal with some spaces $S^{r,b}_{pq}B(\mathbb{Q}^n)$ in the context of Haar bases and Faber bases using specific arguments. This may justify to restrict our attention now to one peculiar point. By Theorem 1.67 (i) the spaces $S^r_{pp}B(\mathbb{Q}^2)$ with $1 \le p \le \infty$ and

$0 < r < M \in \mathbb{N}$ can be equivalently normed by (1.294), which can be rewritten as (1.295). Let $Q = \mathbb{Q}^2$. The natural extension of (1.294) is given by

$$\begin{aligned} \|f\,|S^{r,b}_{pp}B(Q)\|^M &= \|f\,|L_p(Q)\| \\ &+ \Big(\int_0^{1/2} h^{-rp}|\log h|^{bp}\big(\|\Delta^M_{h,1,Q}f\,|L_p(Q)\|^p + \|\Delta^M_{h,2,Q}f\,|L_p(Q)\|^p\big)\frac{\mathrm{d}h}{h}\Big)^{1/p} \\ &+ \Big(\int_0^{1/2}\int_0^{1/2} (h_1h_2)^{-rp}|\log h_1|^{bp}|\log h_2|^{bp}\|\Delta^{M,M}_{h,Q}f\,|L_p(Q)\|^p\frac{\mathrm{d}h}{h_1h_2}\Big)^{1/p} \end{aligned} \tag{1.369}$$

where we used the same notation as there, in particular (1.291)–(1.293). Assuming that (1.332) is an equivalent norm in $B^{s,b}_{pp}(I)$ then one has

$$\begin{aligned} \|f\,|S^{r,b}_{pp}B(Q)\|^M &\sim \Big(\int_0^1 \|f(\cdot,x_2)|B^{r,b}_{pp}(I)\|^p_M\,\mathrm{d}x_2\Big)^{1/p} \\ &+ \Big(\int_0^1\int_0^{1/2} h^{-rp}|\log h|^{bp}\|\Delta^M_{h,2,Q}f(\cdot,x_2)\,|B^{r,b}_{pp}(I)\|^p_M\,\frac{\mathrm{d}h}{h}\,\mathrm{d}x_2\Big)^{1/p}, \end{aligned} \tag{1.370}$$

where $\|\cdot|B^{r,b}_{pp}(I)\|_M$ refers to the second equivalence in (1.332) with r in place of s and with $p = q$. But afterwards one is in the same situation as in the proof of Theorem 1.67. However for this purpose one has first to ask whether the equivalent norms (1.166), (1.167) for the spaces $S^r_{pq}B(\mathbb{R}^2)$ have counterpart for $S^{r,b}_{pq}B(\mathbb{R}^2)$. This is of self-contained interest and we formulate the expected outcome as far as the generalisation of (1.166) is concerned. Let, as there, $0 < r < M \in \mathbb{N}$ and $1 \le p,q \le \infty$. Let $b \in \mathbb{R}$ and let $S^{r,b}_{pq}B(\mathbb{R}^2)$ be the corresponding spaces introduced in Definition 1.79 (i). We use the same notation as in connection with (1.166). Then one may ask whether

$S^{r,b}_{pq}B(\mathbb{R}^2)$ is the collection of all $L_p(\mathbb{R}^2)$ such that

$$\begin{aligned} &\|f\,|L_p(\mathbb{R}^2)\| + \Big(\int_0^{1/2} h^{-rq}|\log h|^{bq}\|\Delta^M_{h,1,Q}f\,|L_p(\mathbb{R}^2)\|^q\,\frac{\mathrm{d}h}{h}\Big)^{1/q} \\ &\quad+ \Big(\int_0^{1/2} h^{-rq}|\log h|^{bq}\|\Delta^M_{h,2,Q}f\,|L_p(\mathbb{R}^2)\|^q\,\frac{\mathrm{d}h}{h}\Big)^{1/q} \\ &\quad+ \Big(\int_0^{1/2}\int_0^{1/2} (h_1h_2)^{-rq}|\log h_1|^{bq}|\log h_2|^{bq}\|\Delta^{M,M}_{h,Q}f\,|L_p(\mathbb{R}^2)\|^q\,\frac{\mathrm{d}h}{h_1h_2}\Big)^{1/q} \end{aligned} \tag{1.371}$$

is finite (equivalent norms).

Maybe one can prove this assertion combining the arguments resulting in [ST87, Theorem 2, p. 122] with the techniques in the papers mentioned in Remark 1.75. Now we can try to follow the proof of Theorem 1.67. If (1.371) with $p = q$ is established as an equivalent norm in $S^{r,b}_{pp}B(\mathbb{R}^2)$ then one has (1.303) with $S^{r,b}_{pp}B(Q)$ in place of $S^r_{pp}B(Q)$ and (1.369). Step 2 of this proof can also be carried over. This follows from (1.370) and the comments about extension operators in Point 8 in Section 1.3.3.

Chapter 2
Haar bases

2.1 Classical theory and historical comments

So far we described wavelet bases in isotropic spaces $A^s_{pq}(\mathbb{R}^n)$, Theorem 1.18, in spaces $S^r_{pq}A(\mathbb{R}^2)$ with dominating mixed smoothness, Theorem 1.54, and their restrictions $S^r_{pp}B(\mathbb{Q}^2)$ to cubes, Theorem 1.66. All these bases originate from Daubechies wavelets and we always assumed that they are sufficiently smooth with (1.69), (1.73) as typical and natural restrictions for the required smoothness u. In this book we are mainly interested in sampling, numerical integration and discrepancy in the context of function spaces. For this purpose the indicated wavelet expansions are occasionally helpful but they are not very effective. We rely on the better adapted Haar bases and Faber bases and develop the corresponding theory in the Chapters 2, 3. Although Haar bases, Faber bases and their numerous descendants have a history of some 100 years they are of vital interest up to our time, maybe now even more than a few decades before. This may justify to have a closer look at the papers of the old masters and to find out what they are telling us nowadays. This provides also a better understanding of what follows.

Let $I = (0,1)$ be the unit interval on $\mathbb{R}$. Let

$$\big\{ h_0,\, h_{jm} : j \in \mathbb{N}_0;\ m = 0, \ldots, 2^j - 1 \big\} \tag{2.1}$$

be the L_∞-normalised orthogonal *Haar system* in $L_2(I)$ where h_0 is the characteristic function of $[0,1)$ and

$$h_{jm}(x) = \begin{cases} 1 & \text{if } 2^{-j}m \le x < 2^{-j}m + 2^{-j-1}, \\ -1 & \text{if } 2^{-j}m + 2^{-j-1} \le x < 2^{-j}(m+1), \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The *Faber system*

$$\big\{ v_0,\, v_1,\, v_{jm} : j \in \mathbb{N}_0;\ m = 0, \ldots, 2^j - 1 \big\} \tag{2.3}$$

is the collection of functions

$$v_0(x) = 1 - x, \quad v_1(x) = x, \quad x \in I, \tag{2.4}$$

and

$$\begin{aligned} v_{jm}(x) &= 2^{j+1} \int_0^x h_{jm}(y)\, \mathrm{d}y \\ &= \begin{cases} 2^{j+1}\big(x - 2^{-j}m\big) & \text{if } 2^{-j}m \le x < 2^{-j}m + 2^{-j-1}, \\ 2^{j+1}\big(2^{-j}(m+1) - x\big) & \text{if } 2^{-j}m + 2^{-j-1} \le x < 2^{-j}(m+1), \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{2.5}$$

Figure 2.1. The Haar system goes back to A. Haar, [Haar10], 1910, whereas the Faber system had been introduced by G. Faber, [Fab09], 1909. We add in Remarks 2.3, 2.4 below a few more detailed historical comments. But first we discuss (and prove partly,

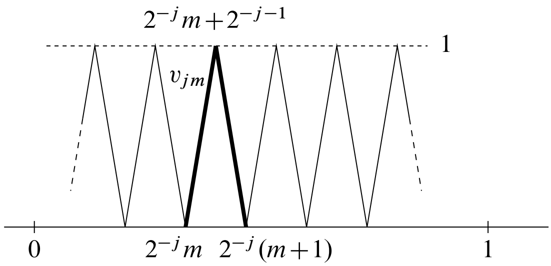

Figure 2.1. Faber system.

following the old masters) some classical fundamental assertions. Recall that $C(I)$ is the space of all complex-valued continuous functions on the closed interval $\bar{I} = [0,1]$ furnished with the L_∞-norm. Let as before

$$(\Delta_h^1 f)(x) = f(x+h) - f(x), \quad \Delta_h^{M+1} f = \Delta_h^1(\Delta_h^M f), \tag{2.6}$$

be the usual differences, $x \in \mathbb{R}$, $h \in \mathbb{R}$, $M \in \mathbb{N}$. In particular,

$$-\tfrac{1}{2}\big(\Delta^2_{2^{-j-1}} f\big)\big(2^{-j}m\big) = f\big(2^{-j}m+2^{-j-1}\big) - \tfrac{1}{2} f\big(2^{-j}m\big) - \tfrac{1}{2} f\big(2^{-j}(m+1)\big). \tag{2.7}$$

Between Definition 1.17 and Theorem 1.18 we recalled what is meant by a *basis*, *unconditional basis* and *conditional basis* in a separable quasi-Banach space.

Theorem 2.1. (i) *The Haar system* (2.1), (2.2) *is*

- *an orthogonal basis in $L_2(I)$,*
- *an unconditional basis in $L_p(I)$ with $1 < p < \infty$,*
- *and a conditional basis in $L_1(I)$.*

Furthermore,

$$f = \int_I f(y)\,\mathrm{d}y + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} 2^j \int_I f(y) h_{jm}(y)\,\mathrm{d}y \; h_{jm}(x), \quad x \in I, \tag{2.8}$$

for any $f \in L_1(I)$.

(ii) *Let $f \in L_1(I)$ and let*

$$f^J(x) = \int_I f(y)\,\mathrm{d}y + \sum_{j=0}^{J} \sum_{m=0}^{2^j-1} 2^j \int_I f(y) h_{jm}(y)\,\mathrm{d}y \; h_{jm}(x), \quad x \in I, \tag{2.9}$$

$J \in \mathbb{N}_0$. *Then*

$$f^J(x) \to f(x) \quad a.e. \text{ if } J \to \infty \tag{2.10}$$

(*pointwise convergence almost everywhere*). *If* $f \in C(I)$ *then*

$$f^J(x) \implies f(x) \quad \text{if } J \to \infty \tag{2.11}$$

(*uniform convergence*).

(iii) *The Faber system* (2.3)–(2.5) *is a conditional basis in* $C(I)$. *Furthermore,*

$$f(x) = f(0)v_0(x) + f(1)v_1(x) - \frac{1}{2}\sum_{j=0}^{\infty}\sum_{m=0}^{2^j-1}\big(\Delta^2_{2^{-j-1}}f\big)\big(2^{-j}m\big)v_{jm}(x) \tag{2.12}$$

for any $f \in C(I)$ *and* $0 \le x \le 1$.

Proof. In Steps 1–4 we prove the classical parts established before 1930, following essentially Haar, Faber and Schauder. In Step 5 we comment on more recent assertions.

Step 1. Orthogonal L_2-basis. Obviously, the functions in (2.1) are pairwise orthogonal. It follows by iteration that the characteristic function

$$\chi_{jm} \quad \text{of} \quad I_{jm} = \big[2^{-j}m, 2^{-j}(m+1)\big), \quad j \in \mathbb{N}_0;\ m = 0, \ldots, 2^j - 1, \tag{2.13}$$

is a finite linear combination of h_0 and h_{kl} with $k < j$. This shows that the Haar system (2.1) is an orthogonal basis in $L_2(I)$ with (2.8).

Step 2. We prove part (ii). Let f^J with $f \in L_1(I)$ be given by (2.9). First we wish to show by induction that

$$f^J(x) = \sum_{m=0}^{2^{J+1}-1} 2^{J+1}\int_{I_{J+1,m}} f(y)\mathrm{d}y\ \chi_{J+1,m}(x), \quad x \in I,\ J \in \mathbb{N}_0. \tag{2.14}$$

From (2.9) with $J = 0$ it follows that

$$\begin{aligned} f^0(x) = &\int_{I_{1,0}} f(y)\mathrm{d}y + \int_{I_{1,1}} f(y)\mathrm{d}y \\ &+ \Big(\int_{I_{1,0}} f(y)\mathrm{d}y - \int_{I_{1,1}} f(y)\mathrm{d}y\mathrm{d}y\Big)\big(\chi_{1,0}(x) - \chi_{1,1}(x)\big) \\ = &\ 2\int_{I_{1,0}} f(y)\mathrm{d}y\ \chi_{1,0}(x) + 2\int_{I_{1,1}} f(y)\mathrm{d}y\ \chi_{1,1}(x), \end{aligned} \tag{2.15}$$

hence (2.14) with $J = 0$. For $J \in \mathbb{N}$ one obtains by (2.9) that

$$\begin{aligned} f^J(x) = f^{J-1}(x) + 2^J \sum_{m=0}^{2^J-1}\Big(&\int_{I_{J+1,2m}} f(y)\mathrm{d}y \\ &- \int_{I_{J+1,2m+1}} f(y)\mathrm{d}y\Big)\big(\chi_{J+1,2m}(x) - \chi_{J+1,2m+1}(x)\big). \end{aligned} \tag{2.16}$$

We insert (2.14) with $J-1$ in place of J and decompose

$$I_{J,m} = I_{J+1,2m} \cup I_{J+1,2m+1} \quad \text{and} \quad \chi_{J,m} = \chi_{J+1,2m} + \chi_{J+1,2m+1}.$$

Then we obtain (2.14). It follows that

$$f(x) - f^J(x) = \sum_{m=0}^{2^{J+1}-1} 2^{J+1} \int_{I_{J+1,m}} \big(f(x) - f(y)\big)\mathrm{d}y \; \chi_{J+1,m}(x). \tag{2.17}$$

Recall that $x \in I$ is called a Lebesgue point of $f \in L_1(I)$ (extended by zero from I to $\mathbb{R}$) if

$$\frac{1}{2r} \int_{|x-y|\le r} |f(x) - f(y)|\mathrm{d}y \to 0 \quad \text{for } r \to 0, \tag{2.18}$$

[Ste70, p. 11]. If $x \in I$ is a Lebesgue point and $x \in I_{J+1,m}$ then it follows from (2.17), (2.18) that

$$\begin{aligned} |f(x) - f^J(x)| &= 2^{J+1}\Big| \int_{I_{J+1,m}} \big(f(x) - f(y)\big)\mathrm{d}y \Big| \\ &\le 2^{J+1} \int_{|x-y|\le 2^{-J}} |f(x) - f(y)|\,\mathrm{d}y \to 0 \end{aligned} \tag{2.19}$$

for $J \to \infty$. This proves (2.10). If $f \in C(I)$ then one obtains (2.11) from (2.19) and the uniform continuity of f.

Step 3. We prove that the Haar system (2.1) is a basis in $L_p(I)$ with $1 \le p < \infty$. By (2.14) and Hölder's inequality one has

$$|f^J(x)| \le \sum_{m=0}^{2^{J+1}-1} 2^{J+1} \Big(\int_{I_{J+1,m}} |f(y)|^p \mathrm{d}x \Big)^{1/p} 2^{-(J+1)(1-\frac{1}{p})} \chi_{J+1,m}(x) \tag{2.20}$$

and

$$\int_I |f^J(x)|^p \mathrm{d}x \le \int_I |f(x)|^p \mathrm{d}x \quad \text{for all } J \in \mathbb{N}_0. \tag{2.21}$$

This can be complemented by

$$\|g^J \,|L_\infty(I)\| \le \|g\,|L_\infty(I)\|, \quad g \in L_\infty(I),\; J \in \mathbb{N}_0. \tag{2.22}$$

Let $f \in L_p(I)$. For any $\varepsilon > 0$ there are two functions g_ε and h_ε with

$$f = g_\varepsilon + h_\varepsilon, \quad g_\varepsilon \in L_\infty(I), \quad \|h_\varepsilon\,|L_p(I)\| \le \varepsilon. \tag{2.23}$$

One obtains by (2.14) and (2.21), (2.22) uniformly in J that

$$\begin{aligned} \|f - f^J\,|L_p(I)\| &\le \|g_\varepsilon - g_\varepsilon^J\,|L_p(I)\| + \|h_\varepsilon - h_\varepsilon^J\,|L_p(I)\| \\ &\le 2\varepsilon + \big(2\,\|g_\varepsilon\,|L_\infty(I)\|\big)^{1-\frac{1}{p}} \Big(\int_I |(g_\varepsilon - g_\varepsilon^J)(x)|\mathrm{d}x \Big)^{1/p}. \end{aligned} \tag{2.24}$$

By (2.22), Lebesgue's bounded convergence theorem and (2.19) it follows that

$$\| f - f^J \,|L_p(I)\| \le 3\varepsilon \quad \text{if } J \ge J(f,\varepsilon). \tag{2.25}$$

Hence $f^J \to f$ in $L_p(I)$. The uniqueness of the representation (2.8) is a consequence of the orthogonality of the system (2.1). Hence (2.1) is a common basis in $L_p(I)$ with $1 \le p < \infty$.

Step 4. We prove that the Faber system (2.3)–(2.5) is a basis in $C(I)$ with the expansion (2.12). Let $f \in C(I)$ and let

$$f_J(x) = f(0)\, v_0(x) + f(1)\, v_1(x) - \frac{1}{2}\sum_{j=0}^{J}\sum_{m=0}^{2^j-1} \big(\Delta^2_{2^{-j-1}} f\big)\big(2^{-j}m\big)\, v_{jm}(x), \tag{2.26}$$

where $J \in \mathbb{N}_0$. We prove by induction that

$$f_J(x) = f(x) \quad \text{if} \quad x = 2^{-(J+1)}m \quad \text{with } m = 0,\ldots,2^{J+1}. \tag{2.27}$$

Let $J = 0$. Then $f_0(0) = f(0)$, $f_0(1) = f(1)$ and by (2.7)

$$f_0(1/2) = \tfrac{1}{2} f(0) + \tfrac{1}{2} f(1) + \big(f(1/2) - \tfrac{1}{2} f(0) - \tfrac{1}{2} f(1)\big) v_{00}(1/2) = f(1/2). \tag{2.28}$$

Assume that for $J \in \mathbb{N}$,

$$f_{J-1}(x) = f(x) \quad \text{if } x = 2^{-J}m \text{ with } m = 0,\ldots,2^J. \tag{2.29}$$

By (2.26) one has

$$f_J(x) = f_{J-1}(x) - \frac{1}{2}\sum_{m=0}^{2^J-1} \big(\Delta^2_{2^{-J-1}} f\big)\big(2^{-J}m\big)\, v_{Jm}(x). \tag{2.30}$$

Since $v_{Jm}(x) = 0$ if $x = 2^{-J}l$ it follows from (2.30), (2.29) that

$$f_J(x) = f_{J-1}(x) = f(x) \quad \text{if } x = 2^{-J}m = 2^{-J-1}2m, \tag{2.31}$$

where $m = 0,\ldots,2^J$. Let $x = 2^{-(J+1)}(2m+1)$ with $m = 0,\ldots,2^J - 1$. Then f_{J-1} is linear in the interval $[2^{-J}m, 2^{-J}(m+1)]$ and it follows from (2.29), (2.30) and (2.7) that

$$\begin{aligned} &f_J\big(2^{-(J+1)}(2m+1)\big) \\ &\quad = \tfrac{1}{2} f\big(2^{-J}m\big) + \tfrac{1}{2} f\big(2^{-J}(m+1)\big) \\ &\qquad + \big[f\big(2^{-J}m + 2^{-J-1}\big) - \tfrac{1}{2} f\big(2^{-J}m\big) - \tfrac{1}{2} f\big(2^{-J}(m+1)\big)\big] v_{Jm}\big(2^{-J}m + 2^{-J-1}\big) \\ &\quad = f\big(2^{-J}m + 2^{-J-1}\big). \end{aligned} \tag{2.32}$$

This proves (2.27). Hence $f_J(x)$ is a piecewise linear function which coincides with $f(x)$ in the nodes $2^{-J-1}m$. Then it follows that

$$f_J(x) \Longrightarrow f(x) \quad \text{if } J \to \infty, \text{convergence in } C(I). \tag{2.33}$$

Hence any $f \in C(I)$ can be represented in $C(I)$ by (2.12). It remains to prove that this expansion is unique. Let

$$0 = a_0 v_0(x) + a_1 v_1(x) + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} a_{jm} v_{jm}(x), \quad x \in I, \tag{2.34}$$

convergence in $C(I)$, where a_0, a_1, a_{jm} are complex numbers. Inserting $x = 0$, $x = 1$ and then $x = 1/2$ one obtains $a_0 = a_1 = 0$ and then $a_{00} = 0$. Iteration gives $a_{jm} = 0$. This proves that the Faber system (2.3) is a basis in $C(I)$.

Step 5. The above arguments reflect the situation at the end of the 1920s. More detailed historical comments will be given below. The above proof does not cover that the Haar basis (2.1) is unconditional in $L_p(I)$ with $1 < p < \infty$ and conditional in $L_1(I)$, and that the Faber basis (2.3) in $C(I)$ is conditional. We comment on these more recent assertions restricting ourselves to references. By [Woj91, p. 62] any basis in $L_1(I)$ is conditional. This applies in particular to the Haar basis (2.1) from which we know by Step 3 that it is a basis in $L_1(I)$. By Step 4 the Faber system (2.3) is a basis in $C(I)$. But any basis in $C(I)$ is conditional, [Woj91, p. 63]. The Haar system (2.1) is an unconditional basis in $L_p(I)$ with $1 < p < \infty$. This is a consequence of the famous Littlewood–Paley theorem for L_p-spaces with $1 < p < \infty$ which will be discussed later on in detail. The assertion itself goes back to J. Marcinkiewicz, [Mar37], who based on [Pal32]. One may also consult [Woj91, Theorem 13, p. 63], [Woj97, Section 8.3] and [LiZ79, Theorem 2.c.5, pp. 155/6]. □

Remark 2.2. Both in (2.8)–(2.10) with respect to the Haar bases and (2.12), (2.26) with respect to the Faber bases we dealt with J-blocks of corresponding series. Within the J-blocks one may assume that the terms are ordered naturally by $m = 0, \dots, 2^J - 1$ and that assertions of type (2.10), (2.11) are refined by inserting additionally m with the same outcome. But any other rearrangement of the terms within the J-blocks gives the same result. This is not totally obvious. But it follows from the arguments in Steps 2 and 4 applied to J and m.

Remark 2.3. The complete orthogonal system (2.1), (2.2) had been introduced 1910 by A. Haar in his PhD-thesis [Haar10, Kapitel III] (Über eine Klasse von orthogonalen Funktionensystemen). Hilbert was the supervisor (Doktorvater). Chapters I (Divergente Reihen) and II (Theorie der Summation) deal with the unsatisfactory situation in connection with pointwise convergence of corresponding expansions of continuous functions in $\bar{I} = [0,1]$. He mentions in particular trigonometrical series,

$$f(x) \sim \sum_{k=-\infty}^{\infty} a_k \, e^{i2\pi kx}, \quad a_k = \int_I f(y) \, e^{-i2\pi ky} \mathrm{d}y, \tag{2.35}$$

$k \in \mathbb{Z}$, $0 \le x \le 1$. It was known at this time that there are continuous functions such that the corresponding trigonometric series do not converge at any point x with $0 \le x \le 1$. Haar even asked: *Gibt es überhaupt ein orthogonales Funktionensystem, das so beschaffen ist, daß jede stetige Funktion auf die Fouriersche Weise in eine gleichmäßig konvergente Reihe entwickelbar ist, die nach Funktionen dieses Systems fortschreitet?* (Does there exists an orthogonal system of functions such that any continuous function can be expanded in the way of Fourier in a uniformly convergent series with respect to this system?) His answer is yes and he introduced in Chapter III in [Haar10] the system (2.1), (2.2) named nowadays after him. Part (ii) of the above theorem is essentially covered by Haar, including (2.11) (as one of the main goals) and also (2.10) for *willkürliche Funktionen* (arbitrary functions). In a somewhat hidden way he relies on (2.14) and even uses Lebesgue points. Nowadays Haar's paper is usually mentioned as the very first and distinguished example of a wavelet system having the multiresolution property. One may consult [T06, Section 1.7] and the references given there. Using Haar's observations, especially (2.14), Schauder proved in 1928, [Scha28], that the Haar system (2.1), (2.2) is a basis in all spaces $L_p(I)$ with $1 \le p < \infty$. This is covered by Step 3 of the above proof where we followed essentially Schauder's arguments. The system (2.3)–(2.5) goes back to Faber, [Fab09]. He proved that this system is a basis in $C(I)$ and that continuous functions can be represented by (2.12). His arguments are similar as in Step 4 of the above proof. But it is the main aim of his paper *die tiefe Kluft, die zwischen den differenzierbaren and bloß stetigen Funktionen besteht zu erhellen* (to shed light on the deep rift between differentiable functions and merely continuous functions). For this purpose he discusses in detail properties of continuous functions (differentiability, rectifiability) represented as

$$f(x) = \sum_{j,m} a_{jm}\, v_{jm}(x), \quad a_{jm} \in \mathbb{R}, \tag{2.36}$$

in dependence on the admitted coefficients a_{jm}. Faber's construction had been rediscovered in 1927 by Schauder, [Scha27], replacing $\{2^{-j}m : j \in \mathbb{N}_0;\, m = 0, \ldots, 2^j - 1\}$ by more general dense sets $\{t_j\} \subset I$. But otherwise the construction of a basis in $C(I)$ is similar using adapted second differences of type (2.7). However the main aim of Schauder's paper [Scha27] is the proof of the famous fixed point theorem named after him: *A continuous map T of a compact convex set K in a Banach space with $T(K) \subset K$ has a fixed point*. Schauder needed for his proof that the Banach space in question has a basis (later on it came out that this is not necessary). For this reason Schauder presented a (short) list of Banach spaces with basis. The first example is $C(I)$ with the indicated construction. Nowadays it is usual to call these piecewise linear bases in $C(I)$ *Faber–Schauder systems* or *Schauder systems*. We use here only the original dyadic version by Faber. This may justify to speak about *Faber systems* (Faber bases).

Remark 2.4. There was always an interest in Haar systems especially on $\mathbb{R}$, $\mathbb{R}^n$, intervals and cubes. This can be seen by the above classical references at least as far as L_p-spaces on intervals are concerned. It is just one of the main aims of this book and in particular of Chapter 2 to deal with Haar systems in the context of the spaces B^s_{pq}, F^s_{pq}. Then we give also more specific and more recent references. The question for Faber bases in spaces of type $A^s_{pq}(I)$ requires that the embedding

$$A^s_{pq}(I) \hookrightarrow C(I) \text{ is continuous.} \tag{2.37}$$

This is a consequence of the needed pointwise evaluation of $f \in A^s_{pq}(I)$ according to (2.12). It excludes $L_p(I)$, $1 \leq p < \infty$, and all spaces $A^s_{pq}(I)$ with $s < 1/p$. Furthermore, the functions v_{jm} in (2.5) are non-negative. In particular the system (2.3) cannot be orthogonal. The usual orthogonalisation procedure of the Faber system results in the Franklin system. This system goes back to Franklin, [Fra28], who proved in 1928 that this orthogonal system is a basis in $C(I)$. Iteration creates higher orthogonal spline systems. This is mainly due to Z. Ciesielski and his co-workers. As far as unconditional orthogonal spline bases in L_p-spaces and a.e. convergence is concerned we refer in particular to [Cie75], [Cie79] where one finds also further references. In other words, in contrast to the Haar system, both the original Faber system, [Fab09], and the Faber–Schauder systems, [Scha27], did not influence the further development directly, but served as the main ingredient of the recent theory of orthogonal spline systems, especially on intervals and cubes. In 1960 Ciesielski used the Faber system for his celebrated assertion that the Hölder spaces $C^s(I)$, $0 < s < 1$, are isomorphic to ℓ_∞, [Cie60]. It fits to the above remarks that he gave in 1963, [Cie63], a new proof now based on orthogonal Franklin systems. Further historical comments may be found in [Woj91, p. 175]. In recent times systems of Faber type have been used in [JoK01] to construct bases in Besov spaces on fractals.

Remark 2.5. As mentioned above it is of interest to know whether an (orthogonal) basis in $L_2(I)$ is also an (unconditional) basis in $L_p(I)$ with $1 \leq p < \infty$ and whether corresponding expansions converge almost everywhere. It was well known at the beginning of the last century that the most distinguished orthonormal basis in $L_2(I)$, the system

$$\left\{ e^{i2\pi kx} : k \in \mathbb{Z} \right\} \tag{2.38}$$

of periodic trigonometrical functions, causes a lot of trouble. This attracted the attention of several outstanding mathematicians. It was one of the motivations of Haar in [Haar10] to look for systems with better properties. It seems to be reasonable to complement the above historically-oriented comments by a few remarks about (2.38). By [Woj91, pp. 40/62] the system (2.38) is a conditional basis in $L_p(I)$, $1 < p < \infty$, $p \neq 2$. The question of what can be said about the pointwise convergence of the partial sums

$$s_n(f)(x) = \sum_{|k| \leq n} \int_0^1 e^{-i2\pi ky} f(y) \mathrm{d}y \, e^{i2\pi kx} \to f(x) \tag{2.39}$$

if $n \to \infty$ was a central problem of real analysis over decades. As mentioned above it was known at the beginning of the last century (in particular to Haar) that there are functions $f \in C(I)$ for which the partial sums $s_n(f)$ do not converge uniformly to f. Luzin conjectured in 1913 that one has for any $f \in C(I)$ an a.e. convergence in (2.39), [Luz13]. On the other hand, young Kolmogorov (age 20) constructed 1923 a function $f \in L_1(I)$ for which $s_n(f)(x)$ diverges a.e., [Kol23], and in [Kol26] (now age 23) a function $f \in L_1(I)$ for which $s_n(f)(x)$ diverges everywhere. Luzin's conjecture was confirmed by Carleson 1966, who proved that one has (2.39) a.e. for any function $f \in L_2(I)$, [Car66]. This was extended shortly afterwards by Hunt in [Hunt68] to $f \in L_p(I)$ with $1 < p < \infty$. A precise formulation may be found in [Gra04, Theorems 3.6.13, 3.6.14, pp. 229/230] with a reference to Chapter 10 of this book for (rather long) proofs.

2.2 Haar bases on $\mathbb{R}$ and on intervals

2.2.1 Introduction and plan of the chapter

Theorem 1.18 is a satisfactory characterisation of the isotropic spaces $A^s_{pq}(\mathbb{R}^n)$ in terms of compactly supported smooth wavelets. By Theorem 1.54 one has a corresponding characterisation in terms of compactly supported wavelets of the spaces $S^r_{pq}A(\mathbb{R}^n)$ ($n = 2$) with dominating mixed smoothness. On the other hand Theorem 2.1 (i) describes classical expansions of $L_p(I)$, and similarly of $L_p(\mathbb{R})$, with $1 < p < \infty$, by Haar bases. This can be strengthened in analogy to (1.74) where $\lambda \in f^0_{p,2}$ reflects the famous Littlewood–Paley theorem (recall that $L_p(\mathbb{R}) = F^0_{p,2}(\mathbb{R})$ and similarly $L_p(I) = F^0_{p,2}(I)$, $1 < p < \infty$). The Haar system in one dimension (on $\mathbb{R}$) can be extended to $\mathbb{R}^n$, $n \geq 2$, either as in (1.59), called *Haar wavelet system*, or as in (1.198), called *Haar tensor system*. What are the natural counterparts of Theorem 1.18 (characterisation of $A^s_{pq}(\mathbb{R}^n)$ in terms of Haar wavelet bases) and of Theorem 1.54 (characterisation of $S^r_{pq}A(\mathbb{R}^n)$ in terms of Haar tensor bases)? This is the main topic of Chapter 2. Section 2.2 deals with Haar bases for $A^s_{pq}(\mathbb{R})$ and $A^s_{pq}(I)$. We obtain in particular a new proof of the Littewood–Paley characterisations of $L_p(\mathbb{R})$ and $L_p(I)$ in terms of the (inhomogeneous) Haar system (2.1). The original classical version relies on the (homogeneous) Haar system consisting of h_{jm} according to (2.2) with $j \in \mathbb{Z}$, $m \in \mathbb{Z}$. This will be done in Section 2.2.5. Section 2.3 deals with (inhomogeneous) Haar wavelet bases in $A^s_{pq}(\mathbb{R}^n)$ (and on cubes) and Section 2.4 with Haar tensor bases in spaces with dominating mixed smoothness. In Section 2.5 we extend some assertions to spline bases.

Beyond the celebrated Littlewood–Paley theory for the L_p-spaces with $1 < p < \infty$ there are only a few assertions in literature about Haar bases in spaces of type A^s_{pq}. Some relevant papers dealing with Haar bases in Besov spaces B^s_{pq} will be mentioned later on. But we have no references about Haar bases in F^s_{pq}-spaces, including (fractional) Sobolev spaces $H^s_p = F^s_{p,2}$. It is the main aim of Chapter 2 to raise the theory of Haar bases for isotropic spaces A^s_{pq} and for spaces $S^r_{pq}A$ with dominating mixed smoothness at the same level as in Theorems 1.18, 1.54 for smooth compactly supported wavelets. This might be of interest for its own sake. But in our context these assertions serve as the fundamentals for a corresponding theory of Faber bases especially in spaces with dominating mixed smoothness which in turn paves the way to deal afterwards with sampling, numerical integration and discrepancy.

2.2.2 Inequalities

Let χ_{jm} be the characteristic function of the interval

$$I_{jm} = \big[2^{-j}m, 2^{-j}(m+1)\big), \quad j \in \mathbb{N}_0,\ m \in \mathbb{Z}, \tag{2.40}$$

on the real line $\mathbb{R}$. Let $\chi^{(p)}_{jm} = 2^{j/p}\,\chi_{jm}$ be its p-normalised modification, $0 < p \le \infty$. Let b_{pq} and f_{pq} with $0 < p,q \le \infty$ be the sequence spaces according to Definition 1.3 with $n = 1$, hence the spaces of all sequences

$$\mu = \big\{\mu_{jm} \in \mathbb{C} : j \in \mathbb{N}_0,\ m \in \mathbb{Z}\big\} \tag{2.41}$$

quasi-normed by

$$\|\mu \,|b_{pq}\| = \Big(\sum_{j=0}^{\infty}\Big(\sum_{m\in\mathbb{Z}} |\mu_{jm}|^p\Big)^{q/p}\Big)^{1/q} \tag{2.42}$$

and

$$\|\mu \,|f_{pq}\| = \Big\|\Big(\sum_{j,m} |\mu_{jm}\,\chi^{(p)}_{jm}(\cdot)|^q\Big)^{1/q} |L_p(\mathbb{R})\Big\|. \tag{2.43}$$

The convergence of the series below must be understood as in Theorem 1.7 and as in the references mentioned in Remark 1.8. We will not stress this point in the sequel.

Proposition 2.6. (i) *Let* $0 < p,q \le \infty$ *and*

$$\max\Big(\frac{1}{p}, 1\Big) - 1 < s < \frac{1}{p}. \tag{2.44}$$

Then

$$f = \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}} \mu_{jm}\, 2^{-j(s-\frac{1}{p})}\,\chi_{jm}, \quad \mu \in b_{pq}, \tag{2.45}$$

belongs to $B^s_{pq}(\mathbb{R})$ *and*

$$\|f \,|B^s_{pq}(\mathbb{R})\| \le c\, \|\mu \,|b_{pq}\| \tag{2.46}$$

for some $c > 0$ *and all* $\mu \in b_{pq}$.

(ii) *Let* $0 < p,q < \infty$ *and*

$$\max\Big(\frac{1}{p}, \frac{1}{q}, 1\Big) - 1 < s < \min\Big(\frac{1}{p}, \frac{1}{q}\Big). \tag{2.47}$$

Then

$$f = \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}} \mu_{jm}\, 2^{-j(s-\frac{1}{p})}\,\chi_{jm}, \quad \mu \in f_{pq}, \tag{2.48}$$

belongs to $F^s_{pq}(\mathbb{R})$ *and*

$$\|f\,|F^s_{pq}(\mathbb{R})\| \le c\,\|\mu\,|f_{pq}\| \tag{2.49}$$

for some $c>0$ *and all* $\mu \in f_{pq}$.

Proof. Step 1. Let ψ_F and ψ_M be real compactly supported wavelets on $\mathbb{R}$ according to (1.55), (1.56) where we always assume that $u \in \mathbb{N}$ is sufficiently large. Let the L_2-norms of ψ_M and ψ_M be 1. Then

$$\big\{\psi_F(x-l),\, 2^{k/2}\psi_M(2^k x-l) \,:\, k \in \mathbb{N}_0,\ l \in \mathbb{Z}\big\} \tag{2.50}$$

is an orthonormal basis in $L_2(\mathbb{R})$. The function χ_{jm} can be represented as

$$\chi_{jm}(x) = \sum_{l\in\mathbb{Z}} \lambda_l^{0,F}(\chi_{jm})\,\psi_F(x-l) + \sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}} \lambda_l^{k,M}(\chi_{jm})\psi_M(2^k x-l) \tag{2.51}$$

with

$$\lambda_l^{0,F}(\chi_{jm}) = \int_{\mathbb{R}} \chi_{jm}(y)\,\psi_F(y-l)\,\mathrm{d}y, \quad l \in \mathbb{Z}, \tag{2.52}$$

and

$$\lambda_l^{k,M}(\chi_{jm}) = 2^k \int_{\mathbb{R}} \chi_{jm}(y)\,\psi_M(2^k y-l)\,\mathrm{d}y, \quad k \in \mathbb{N}_0,\ l \in \mathbb{Z}. \tag{2.53}$$

We concentrate on $\lambda_l^k(\chi_{jm}) = \lambda_l^{k,M}(\chi_{jm})$. The terms with $\lambda_l^{0,F}(\chi_{jm})$ can always be incorporated appropriately. One has for some $c>0$ and all admitted k,l,j,m that

$$|\lambda_l^k(\chi_{jm})| \le c\,\min\big(1, 2^{-(j-k)}\big). \tag{2.54}$$

Furthermore,

$$\lambda_l^k(\chi_{jm}) = 0 \text{ if } \operatorname{supp}\psi_M(2^k\cdot -l) \subset I_{jm} \text{ or } \operatorname{supp}\psi_M(2^k\cdot -l) \subset \mathbb{R}\setminus I_{jm}. \tag{2.55}$$

In particular, if $k>j$ then only terms with $l \sim 2^{k-j}m$ and $l \sim 2^{k-j}(m+1)$ are of interest. We insert (2.51) in (2.45), (2.48) and order the outcome with respect to $\psi_F(x-l)$ and $\psi_M(2^k x-l)$. We concentrate on typical terms indicating similar terms by +++ (including the terms with $\psi_F(x-l)$). Then

$$f = f_1 + f_2 \tag{2.56}$$

where

$$f_1 = \sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}} \nu^1_{kl}\,2^{-k(s-\frac{1}{p})}\,\psi_M(2^k x-l) + ++ \tag{2.57}$$

with

$$\nu^1_{kl} = \sum_{j\ge k}\sum_m \lambda_l^k(\chi_{jm})\,2^{-(j-k)(s-\frac{1}{p})}\mu_{jm} \tag{2.58}$$

and

$$f_2 = \sum_{k=1}^{\infty} \sum_{l \in \mathbb{Z}} \nu_{kl}^2 \, 2^{-k(s-\frac{1}{p})} \, \psi_M(2^k x - l) + ++ \tag{2.59}$$

with

$$\nu_{kl}^2 = \sum_{j<k} \lambda_l^k \big(\chi_{j,2^{j-k}l}\big) \, 2^{-(j-k)(s-\frac{1}{p})} \, \mu_{j,2^{j-k}l}. \tag{2.60}$$

Here (2.60) relies on the observation (2.55) that only terms with $l \sim 2^{k-j}m$ are of interest written now as $m = 2^{j-k}l$ (and terms +++) in the understanding that $\chi_{j,t} = \chi_{j,t}$ if $t \in \mathbb{Z}$ and $\chi_{j,t} = 0$ otherwise (similarly for $\mu_{j,t}$). Furthermore, in (2.58) the summation over m can be written as

$$\sum_m = \sum_{|2^{-j}m - 2^{-k}l| \le c2^{-k}} = \sum_{m \in N(j-k,l)} \tag{2.61}$$

where $N(j-k,l)$ indicates the corresponding blocks of $2^{j-k}\mathbb{Z}$ in which $\mathbb{Z}$ is decomposed.

Step 2. First we deal with the f_1-terms in case of B-spaces. One has by (2.58), (2.54) that

$$|\nu_{kl}^1| \le c \sum_{j \ge k} \sum_{m \in N(j-k,l)} 2^{-(j-k)(s-\frac{1}{p}+1)} \, |\mu_{jm}|. \tag{2.62}$$

If $p \le 1$ then one obtains

$$\begin{aligned} \sum_{l \in \mathbb{Z}} |\nu_{kl}^1|^p &\le c' \sum_{j \ge k} 2^{-(j-k)(s-\frac{1}{p}+1)p} \sum_{l \in \mathbb{Z}} \sum_{m \in N(j-k,l)} |\mu_{jm}|^p \\ &\le c' \sum_{j \ge k} 2^{-(j-k)(s-\frac{1}{p}+1)p} \sum_{m \in \mathbb{Z}} |\mu_{jm}|^p. \end{aligned} \tag{2.63}$$

Since $s > \frac{1}{p} - 1$ one obtains by a standard argument (explicitly formulated in [Ryc99a, Lemma 2, p. 284]) that

$$\|\nu^1 \,|b_{pq}\| \le c \, \|\mu \,|b_{pq}\|. \tag{2.64}$$

Let $p > 1$. The set $N(j-k,l)$ has $\sim 2^{j-k}$ terms. Then one has by (2.62) and Hölder's inequality that

$$\begin{aligned} |\nu_{kl}^1| &\le c \sum_{j \ge k} 2^{-(j-k)(s-\frac{1}{p}+1)} \, 2^{(j-k)(1-\frac{1}{p})} \Big(\sum_{m \in N(j-k,l)} |\mu_{jm}|^p \Big)^{1/p} \\ &\le c \sum_{j \ge k} 2^{-(j-k)s} \Big(\sum_{m \in N(j-k,l)} |\mu_{jm}|^p \Big)^{1/p}. \end{aligned} \tag{2.65}$$

Then it follows for any $\varepsilon > 0$ that

$$\sum_{l \in \mathbb{Z}} |\nu_{kl}^1|^p \le c_\varepsilon \sum_{j \ge k} 2^{-(j-k)(s-\varepsilon)p} \sum_{m \in \mathbb{Z}} |\mu_{jm}|^p. \tag{2.66}$$

Since $s > 0$ one obtains (2.64). Next we deal with the f_1-terms in case of F-spaces. Let $0 < w < \min(1, p, q)$ and let M be the classical Hardy–Littlewood maximal function. Then

$$\Big\| \Big(\sum_{j,m} \big(M |g_{jm}|^w \big)(\cdot)^{q/w} \Big)^{1/q} | L_p(\mathbb{R}) \Big\| \le c \, \Big\| \Big(\sum_{j,m} |g_{jm}(\cdot)|^q \Big)^{1/q} | L_p(\mathbb{R}) \Big\| \tag{2.67}$$

is the well-known vector-valued maximal inequality. Explanations and references may be found in [T06, p. 384]. For $x \in I_{kl}$ one has

$$\begin{aligned} \Big(\chi_{kl}(x) \sum_{m \in N(j-k,l)} |\mu_{jm}| \Big)^w &\le \chi_{kl}(x) \sum_{m \in N(j-k,l)} |\mu_{jm}|^w \\ &= 2^j \int_{\mathbb{R}} \sum_{m \in N(j-k,l)} |\mu_{jm}|^w \, \chi_{jm}(y) \, \mathrm{d}y \\ &\le c \, 2^{j-k} \, M \Big(\sum_{m \in N(j-k,l)} |\mu_{jm}|^w \, \chi_{jm}(\cdot) \Big)(x). \end{aligned} \tag{2.68}$$

Then one obtains by (2.62) that

$$\begin{aligned} &|\nu^1_{kl}| \, \chi^{(p)}_{kl}(x) \\ &\quad \le \sum_{j \ge k} 2^{-(j-k)(s - \frac{1}{p} + 1)} \, 2^{-\frac{j-k}{p}} \, 2^{\frac{j-k}{w}} M \Big(\sum_{m \in N(j-k,l)} |\mu_{jm}|^w \, \chi^{(p)}_{jm}(\cdot)^w \Big)(x)^{1/w}. \end{aligned} \tag{2.69}$$

Recall that for fixed $j \in \mathbb{N}_0$ the characteristic functions χ_{jm} have disjoint supports. Then it follows for $\varepsilon > 0$ that

$$\begin{aligned} &\sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}} |\nu^1_{kl} \, \chi^{(p)}_{kl}(x)|^q \\ &\quad \le c_\varepsilon \sum_{k,l} \sum_{j \ge k} 2^{-(j-k)(s + 1 - \frac{1}{w} - \varepsilon) q} \, M \Big(\sum_{m \in N(j-k,l)} |\mu_{jm}|^w \, \chi^{(p)}_{jm}(\cdot)^w \Big)(x)^{q/w}. \end{aligned} \tag{2.70}$$

The counterpart

$$\|\nu^1 \, | f_{pq}\| \le c \, \|\mu \, | f_{pq}\| \tag{2.71}$$

of (2.64) with f_{pq} according to (2.43) in place of b_{pq} follows now from (2.47), w and ε chosen such that

$$s > \varepsilon + \tfrac{1}{w} - 1, \tag{2.72}$$

(2.67) and the above arguments (referring again to [Ryc99a, Lemma 2, p. 284]).

Step 3. We deal with the f_2-terms in (2.56), (2.59), (2.60). With $j = k - t$ in (2.60) one obtains by (2.54) that

$$|\nu^2_{kl}| \le c \sum_{t=1}^{k} 2^{t(s - \frac{1}{p})} \, |\mu_{k-t, 2^{-t} l}| \tag{2.73}$$

with the indicated agreement after (2.60). Since $s < 1/p$ in case of B-spaces one can argue as above and obtains that

$$\|\nu^2 \,|b_{pq}\| \le c\, \|\mu \,|b_{pq}\|. \tag{2.74}$$

In case of F-spaces we have

$$\begin{aligned} |\nu^2_{kl}|\, 2^{k/p} \chi_{kl}(x) &\le c \sum_{t=1}^{k} |\mu_{k-t,2^{-t}l}|\, 2^{\frac{k-t}{p}} \chi_{kl}(x)\, 2^{ts} \\ &\le c \sum_{t=1}^{k} |\mu_{k-t,2^{-t}l}|\, 2^{\frac{k-t}{p}} \chi_{k-t,2^{-t}l}(x)\, 2^{ts} \end{aligned} \tag{2.75}$$

where we used in the second line (which is a crude estimate) the above agreement. If $s < 0$ then one can argue by the above scheme and one obtains that

$$\|\nu^2 \,|f_{pq}\| \le c\, \|\mu \,|f_{pq}\|, \quad s < 0. \tag{2.76}$$

We wish to extend (2.76) to $s < \min(\frac{1}{p}, \frac{1}{q})$. For this purpose we modify (2.60) by

$$\tilde{\nu}^2_{kl} = \sum_{j<k} \lambda^k_l \big(\chi_{j,2^{j-k}l}\big)\, \tilde{\mu}_{j,2^{j-k}l} \tag{2.77}$$

asking for which s, p, q one obtains a linear and bounded operator from the sequence space f^s_{pq}, quasi-normed by

$$\|\tilde{\mu} \,|f^s_{pq}\| = \Big\| \Big(\sum_{j,m} |\tilde{\mu}_{jm}\, \chi_{jm}(\cdot)|^q \, 2^{jsq} \Big)^{1/q} |L_p(\mathbb{R})\Big\| \tag{2.78}$$

into itself, hence

$$\|\tilde{\nu}^2 \,|f^s_{pq}\| \le c\, \|\tilde{\mu} \,|f^s_{pq}\|. \tag{2.79}$$

With

$$\tilde{\nu}^2_{kl} = 2^{-k(s-\frac{1}{p})} \nu^2_{kl} \quad \text{and} \quad \tilde{\mu}_{j,2^{j-k}l} = 2^{-j(s-\frac{1}{p})} \mu_{j,2^{j-k}l} \tag{2.80}$$

one has (2.60), and (2.79) can be written as

$$\|\nu^2 \,|f_{pq}\| \le c\, \|\mu \,|f_{pq}\|. \tag{2.81}$$

By (2.74) and (2.76) we have (2.81) and hence (2.79) if

$$\begin{cases} 0 < p = q < \infty, & s < 1/p, \\ 0 < p < \infty, 0 < q < \infty, & s < 0. \end{cases} \tag{2.82}$$

We wish to apply the complex interpolation for quasi-Banach spaces according to [MeM00], [KMM07] described in Section 1.1.5. We refer also to pages 71/72, 272 in [T06] where one finds details how to use this interpolation for function spaces and related sequence spaces. We have in particular

$$f^s_{pq} = \big[f^{s_0}_{p_0 q_0}, f^{s_1}_{p_1 p_1} \big]_\theta, \quad 0 < \theta < 1, \tag{2.83}$$

where

$$s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{p_1}, \tag{2.84}$$

and

$$0 < p_0 < \infty,\ 0 < q_0 < \infty,\ s_0 < 0 \quad \text{and} \quad 0 < p_1 < \infty,\ s_1 < 1/p_1, \tag{2.85}$$

are cases covered by (2.82). As indicated in Figure 2.2 we choose a sequence

$$s < s^j_1 < \frac{1}{p^j_1} \to s \quad \text{if } j \to \infty \tag{2.86}$$

and afterwards θ^j and p^j_0 such that for some $s_0 < 0$,

$$s = (1-\theta^j)s_0 + \theta^j s^j_1, \quad \frac{1}{p} = \frac{1-\theta^j}{p^j_0} + \frac{\theta^j}{p^j_1}. \tag{2.87}$$

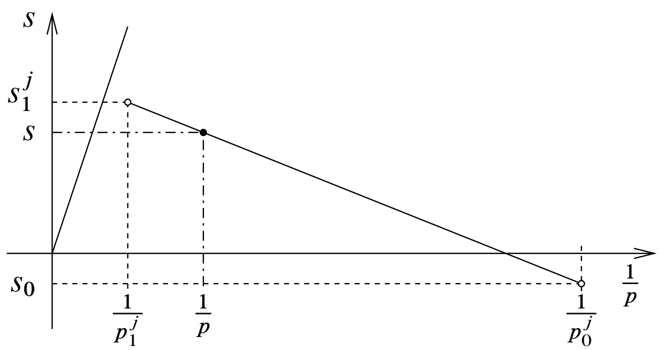

Figure 2.2. Complex interpolation.

Then $\theta^j \to 1$ and

$$\frac{1}{q^j} = \frac{1-\theta^j}{q_0} + \frac{\theta^j}{p^j_1} = s + \varepsilon^j + \frac{1-\theta^j}{q_0} \tag{2.88}$$

with $\varepsilon^j \to 0$ if $j \to \infty$. For fixed j it follows from $q_0 \to 0$ and $q_0 \to \infty$ that any $\frac{1}{q} > s$ can be reached. Hence,

$$\|\tilde{\nu}^2 \,|\, f^s_{pq}\| \le c\, \|\tilde{\mu} \,|\, f^s_{pq}\|, \quad 0 < p, q < \infty,\ s < \min\big(\tfrac{1}{p}, \tfrac{1}{q}\big). \tag{2.89}$$

According to (2.79)–(2.81) this can be written as

$$\|\nu^2 \,|f_{pq}\| \le c\, \|\mu \,|f_{pq}\|, \quad 0 < p,q < \infty,\ s < \min\big(\tfrac{1}{p}, \tfrac{1}{q}\big), \tag{2.90}$$

with ν^2 as in (2.60).

Step 4. After these preparations one can now prove both parts of the proposition as follows. We apply Theorem 1.7 to the atomic decomposition (2.56)–(2.60) of f given by (2.45), (2.48) always assuming that ψ_M is sufficiently smooth and satisfies sufficiently many moment conditions. By (2.44) it follows from (2.64), (2.74) that

$$\|\nu^1 \,|b_{pq}\| + \|\nu^2 \,|b_{pq}\| \le c\, \|\mu \,|b_{pq}\|, \quad 0 < p,q \le \infty, \tag{2.91}$$

incorporating now $p = \infty$ and/or $q = \infty$ in the standard way. This proves part (i) of the proposition. For $0 < p,q < \infty$ and s in (2.47) we choose $\varepsilon > 0$ and $0 < w < \min(p,q,1)$ with (2.72). Then one can apply (2.71), (2.90) and part (ii) of the proposition follows again from Theorem 1.7. □

Remark 2.7. Let I_{jm} be the intervals according to (2.40) and let $K \in \mathbb{N}$. Then one can replace χ_{jm} in (2.45), (2.48) by polynomials P_{jm} in I_{jm} with

$$\text{degree } P_{jm} \le K \quad \text{and} \quad \sup_{x \in I_{jm}} |P_{jm}(x)| \le 1. \tag{2.92}$$

One has again (2.46) for $0 < p,q \le \infty$ and s as in (2.44), and (2.49) for $0 < p,q < \infty$ and s as in (2.47). This follows from the above proof where one has to choose wavelets ψ_M satisfying sufficiently many moment conditions, now in addition larger than K.

We need a second preparation. We extend the Haar system (2.1), (2.2) from I to $\mathbb{R}$ using now standard wavelet notation. Let

$$h_M(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \le x < 1, \\ 0 & \text{if } x \notin [0,1), \end{cases} \tag{2.93}$$

$$h_F(x) = |h_M(x)|, \quad h_{-1,m}(x) = \sqrt{2}\, h_F(x-m), \quad m \in \mathbb{Z}, \tag{2.94}$$

and

$$h_{jm}(x) = h_M\big(2^j x - m\big), \quad j \in \mathbb{N}_0,\ m \in \mathbb{Z}. \tag{2.95}$$

Let $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$. Then

$$\big\{h_{jm} : j \in \mathbb{N}_{-1},\ m \in \mathbb{Z}\big\} \tag{2.96}$$

is an (L_∞-normalised) Haar basis in $L_2(\mathbb{R})$. Let

$$k_{jm}(f) = \int_{\mathbb{R}} k_{jm}(y)\, f(y)\,\mathrm{d}y, \quad k_{jm}(y) = 2^j\, h_{jm}(y), \quad j \in \mathbb{N}_{-1},\ m \in \mathbb{Z}, \tag{2.97}$$

be the obviously adapted local means according to Definitions 1.9, 1.13 with $A = 0$ and $B = 1$. For $s \in \mathbb{R}$ and $0 < p \le \infty$ one has (at least in $L_2(\mathbb{R})$)

$$f = \sum_{j,m} k_{jm}(f)\, h_{jm} = \sum_{j,m} \mu_{jm}(f)\, 2^{-j(s-\frac{1}{p})}\, h_{jm} \tag{2.98}$$

with

$$\mu_{jm}(f) = 2^{j(s-\frac{1}{p})}\, k_{jm}(f), \quad j \in \mathbb{N}_{-1},\ m \in \mathbb{Z}. \tag{2.99}$$

This is the same normalisation as in (2.45), (2.48). We adapt (2.41)–(2.43). Let b^-_{pq} and f^-_{pq} with $0 < p, q \le \infty$ be the spaces of sequences

$$\mu = \big\{ \mu_{jm} \in \mathbb{C} : j \in \mathbb{N}_{-1},\ m \in \mathbb{Z} \big\} \tag{2.100}$$

quasi-normed by

$$\|\mu \,|b^-_{pq}\| = \Big(\sum_{j=-1}^{\infty} \Big(\sum_{m\in\mathbb{Z}} |\mu_{jm}|^p \Big)^{q/p} \Big)^{1/q} \tag{2.101}$$

and

$$\|\mu \,|f^-_{pq}\| = \Big\| \Big(\sum_{j,m} |\mu_{jm}\, \chi^{(p)}_{jm}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}) \Big\| \tag{2.102}$$

where $\chi^{(p)}_{jm}(x) = 2^{j/p}\, \chi_{jm}(x)$ with the characteristic function χ_{jm} of I_{jm} in (2.40) now for $j \in \mathbb{N}_{-1}$ and $m \in \mathbb{Z}$.

Proposition 2.8. (i) *Let* $0 < p, q \le \infty$ *and*

$$\max\Big(\frac{1}{p}, 1\Big) - 1 < s < 1. \tag{2.103}$$

Let $\mu_{jm}(f)$ *be as in* (2.99). *Then there is a* $c > 0$ *such that*

$$\|\mu(f)\,|b^-_{pq}\| \le c\, \|f \,|B^s_{pq}(\mathbb{R})\| \quad \textit{for all } f \in B^s_{pq}(\mathbb{R}). \tag{2.104}$$

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$ *and*

$$\max\Big(\frac{1}{p}, \frac{1}{q}, 1\Big) - 1 < s < 1. \tag{2.105}$$

Let $\mu_{jm}(f)$ *be as in* (2.99). *Then there is a* $c > 0$ *such that*

$$\|\mu(f)\,|f^-_{pq}\| \le c\, \|f \,|F^s_{pq}(\mathbb{R})\| \quad \textit{for all } f \in F^s_{pq}(\mathbb{R}). \tag{2.106}$$

Proof. This follows from Theorem 1.15 with $A = 0$ and $B = 1$ where σ_p and σ_{pq} are given by (1.35) with $n = 1$ and (2.99), (2.100). □

2.2.3 Haar bases on $\mathbb{R}$

Theorem 1.18 gives a satisfactory description of smooth wavelet bases in spaces $A^s_{pq}(\mathbb{R}^n)$. We complement these assertions in Chapter 2 by Haar bases and in Chapter 3 by Faber bases. First we deal in this Section 2.2.3 with Haar bases on $\mathbb{R}$. We rely on the two crucial Propositions 2.6, 2.8 of the preceding Section 2.2.2. As usual we say that a series converges locally in some space $A^\sigma_{pq}(\mathbb{R})$ if it converges in $A^\sigma_{pq}(I)$ for any bounded interval I in $\mathbb{R}$. Otherwise the Haar system $\{h_{jm}\}$ in (2.96) and the sequence spaces b^-_{pq} and f^-_{pq} in (2.100)–(2.102) have the same meaning as before.

Theorem 2.9. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, *and*

$$\frac{1}{p} - 1 < s < \min\Big(1, \frac{1}{p}\Big). \tag{2.107}$$

Let $f \in S'(\mathbb{R})$. *Then* $f \in B^s_{pq}(\mathbb{R})$ *if, and only if, it can be represented as*

$$f = \sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} \mu_{jm}\, 2^{-j(s-\frac{1}{p})}\, h_{jm}, \quad \mu \in b^-_{pq}, \tag{2.108}$$

unconditional convergence being in $S'(\mathbb{R})$ *and locally in any space* $B^\sigma_{pq}(\mathbb{R})$ *with* $\sigma < s$. *The representation* (2.108) *is unique,*

$$\mu_{jm} = \mu_{jm}(f) = 2^{j(s-\frac{1}{p}+1)} \int_{\mathbb{R}} f(x)\, h_{jm}(x)\, \mathrm{d}x, \quad j \in \mathbb{N}_{-1},\ m \in \mathbb{Z}, \tag{2.109}$$

and

$$J : f \mapsto \mu(f), \tag{2.110}$$

is an isomorphic map of $B^s_{pq}(\mathbb{R})$ *onto* b^-_{pq}. *If, in addition,* $p < \infty$, $q < \infty$, *then*

$$\big\{2^{-j(s-\frac{1}{p})}\, h_{jm} : j \in \mathbb{N}_{-1},\ m \in \mathbb{Z}\big\} \tag{2.111}$$

is an unconditional (normalised) basis in $B^s_{pq}(\mathbb{R})$.

(ii) *Let*

$$\begin{cases} 0 < p < \infty,\ 0 < q < \infty, & \max\big(\frac{1}{p}, \frac{1}{q}, 1\big) - 1 < s < \min\big(\frac{1}{p}, \frac{1}{q}, 1\big), \\ 1 < p < \infty,\ 1 < q < \infty, & s = 0, \\ 1 < p < \infty,\ 1 < q \le \infty, & \max\big(\frac{1}{p}, \frac{1}{q}\big) - 1 < s < 0. \end{cases} \tag{2.112}$$

Let $f \in S'(\mathbb{R})$. *Then* $f \in F^s_{pq}(\mathbb{R})$ *if, and only if, it can be represented as*

$$f = \sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} \mu_{jm}\, 2^{-j(s-\frac{1}{p})}\, h_{jm}, \quad \mu \in f^-_{pq}, \tag{2.113}$$

unconditional convergence being in $S'(\mathbb{R})$ *and locally in any spaces* $F^\sigma_{pq}(\mathbb{R})$ *with* $\sigma < s$. *The representation* (2.113) *is unique with* $\mu_{jm}(f)$ *as in* (2.109). *Furthermore,* (2.110) *is an isomorphic map of* $F^s_{pq}(\mathbb{R})$ *onto* f^-_{pq}. *If, in addition,* $q < \infty$ *then* (2.111) *is an unconditional (normalised) basis in* $F^s_{pq}(\mathbb{R})$.

Proof. Step 1. All technicalities such as unconditional convergence, uniqueness, isomorphic maps, duality, in particular the correct interpretation of (2.109) as dual pairing, unconditional basis, are the same as in Step 1 of the proof of [T08, Theorem 1.20, p. 15/16] with a reference to [T06, Section 3.1.3, Theorem 3.5, pp. 153–156]. This will not be repeated here. One may also consult the comments between Definition 1.17 and Theorem 1.18.

Step 2. Let p, q, s be as in the first line in (2.112), in particular $s > 0$. Let f be given by (2.113). Then one obtains from Proposition 2.6 (ii) that $f \in F^s_{pq}(\mathbb{R})$ with (2.49) and f^-_{pq} in place of f_{pq}. Conversely, if $f \in F^s_{pq}(\mathbb{R})$ then one has (2.106). If, in addition, f can be represented by (2.113) with $\mu_{jm} = \mu_{jm}(f)$ according to (2.109) then one obtains by the above arguments that

$$\|f \,|F^s_{pq}(\mathbb{R})\| \sim \|\mu(f)\,| f^-_{pq}\|. \tag{2.114}$$

But (2.113) with $\mu(f)$ as in (2.109) is the same as in (2.97)–(2.99). Recall that $\{h_{jm}\}$ in (2.96) is an orthogonal basis in $L_2(\mathbb{R})$. As detailed in the above references [T06], [T08] the rest is now a matter of completion, embedding and duality according to Theorem 1.20. This proves part (ii) in case of $s > 0$.

Step 3. We restrict the first line in (2.112) to

$$1 < p < \infty, \quad 1 \le q < \infty, \quad 0 < s < \min\left(\frac{1}{p}, \frac{1}{q}\right), \tag{2.115}$$

and apply the duality assertion (1.77). One has

$$1 < p' < \infty, \quad 1 < q' \le \infty, \quad \max\left(\frac{1}{p'} - 1, \frac{1}{q'} - 1\right) < -s < 0. \tag{2.116}$$

Recall again that $\{h_{jm}\}$ is an L_∞-normalised orthogonal basis in $L_2(\mathbb{R})$. The dual of Step 2, restricted to (2.115) proves part (ii) for $s < 0$ (the third line in (2.112)). As for technicalities we refer again to [T06], [T08]. This applies also to the remaining case $1 < p, q < \infty$, $s = 0$, as an outgrowth of the complex interpolation

$$F^0_{pq}(\mathbb{R}) = \left[F^\varepsilon_{pq}(\mathbb{R}), F^{-\varepsilon}_{pq}(\mathbb{R})\right]_{1/2}, \tag{2.117}$$

$\varepsilon > 0$ small, $1 < p, q < \infty$.

Step 4. One can prove part (i) in the same way using (1.76) with $1 \le p, q < \infty$ and in addition the real interpolation

$$\left(B^{s_0}_{pq_0}(\mathbb{R}), B^{s_1}_{pq_1}(\mathbb{R})\right)_{\theta,q} = B^s_{pq}(\mathbb{R}), \quad 0 < \theta < 1, \tag{2.118}$$

where $0 < p, q_0, q_1, q \le \infty$, $s_0 < s_1$, $s = (1-\theta)s_0 + \theta s_1$. □

Remark 2.10. Theorem 2.9 complements the one-dimensional case of Theorem 1.18 where we dealt with compactly supported *smooth* Daubechies wavelets. In [T06, Theorem 1.58, Remark 1.59, pp. 29/30] and [T08, Theorem 2.40, Remark 2.41] we discussed under which circumstances (n-dimensional) Haar systems are bases in some spaces $B^s_{pq}(\mathbb{R}^n)$. In case of $n = 1$ this can now be extended as follows. If $0 < p, q < \infty$ and if s is restricted by (2.107) then the Haar system (2.111) is an unconditional basis in $B^s_{pq}(\mathbb{R})$ with the indicated isomorphic map onto b^-_{pq}, Figure 2.3. On the other hand,

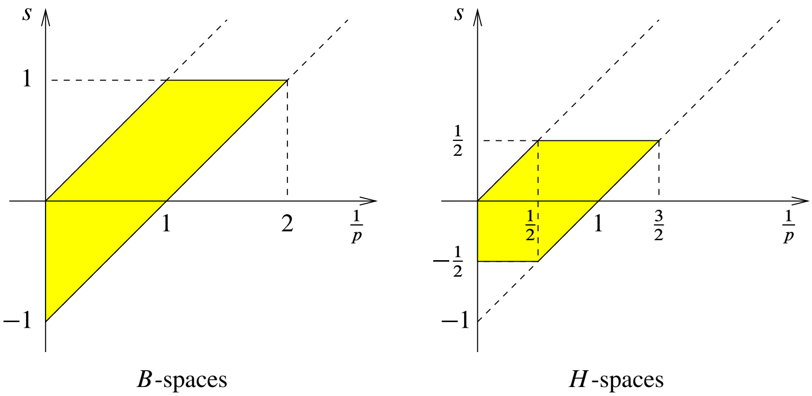

Figure 2.3. Haar bases, $n = 1$.

according to Proposition 2.6 characteristic functions of bounded intervals and hence also Haar functions, are elements of all spaces

$$A^s_{pq}(\mathbb{R}), \quad 0 < p < \infty, \quad 0 < q \le \infty, \quad -\infty < s < 1/p. \tag{2.119}$$

This applies in particular to spaces with $\frac{1}{p} - 1 < s < \frac{1}{p}$ as indicated in Figure 2.3. However if $s > 1$ then the finite linear combinations of Haar functions are not dense in these spaces. This has been mentioned in [T06, Remark 1.59, pp. 29/30] with a reference to [Tri78, Corollary 2, p. 338], based on [Tri73c]. One may consult also the related assertions in [T78]. Also the other restrictions for p, q, s in part (i) of the above theorem are natural in case of B-spaces, [T06, Theorem 1.58, Remark 1.59, pp. 29/30]. Haar bases on homogeneous Besov spaces $\dot{B}^s_{pq}(\mathbb{R})$ and also $\dot{B}^s_{pq}(\mathbb{R}^n)$ have been considered in [Bou95, Theorem 5, p. 503]. Further references will be given in Remark 2.14 below.

Remark 2.11. The proof of the Littlewood–Paley theorem

$$L_p(\mathbb{R}) = F^0_{p,2}(\mathbb{R}), \quad 1 < p < \infty, \tag{2.120}$$

(1.12), can be based on the smooth resolution of unity according to (1.7). This is an application of a smooth Michlin–Hörmander Fourier multiplier theorem in $L_p(\mathbb{R}, \ell_2)$ (and more general in $L_p(\mathbb{R}^n, \ell_2)$, $n \in \mathbb{N}$), $1 < p < \infty$, as it may be found in [T78, Theorem 2.3.3, p. 177]. Based on this observation we obtained in part (ii) of the above theorem as a special case a new proof of one of the most spectacular assertions of real analysis, the *Littlewood–Paley characterisation of* $L_p(\mathbb{R})$*,* $1 < p < \infty$*, in terms of the Haar system*

$$\{h_{jm} : j \in \mathbb{N}_{-1},\ m \in \mathbb{Z}\} : \tag{2.121}$$

By part (ii) of the above theorem and (2.120) the Haar system $\{h_{jm}\}$ is an unconditional basis in $L_p(\mathbb{R})$, $1 < p < \infty$,

$$f = \sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} \lambda_{jm} h_{jm} \tag{2.122}$$

with

$$\lambda_{jm} = 2^j \int_{\mathbb{R}} f(x)\, h_{jm}(x)\, \mathrm{d}x, \quad j \in \mathbb{N}_{-1},\ m \in \mathbb{Z}, \tag{2.123}$$

and

$$\|f\,|L_p(\mathbb{R})\| \sim \Big\| \Big(\sum_{j,m} |\lambda_{jm}\, \chi_{jm}(\cdot)|^2 \Big)^{1/2} |L_p(\mathbb{R}) \Big\| \tag{2.124}$$

where again χ_{jm} is the characteristic function of the interval

$$I_{jm} = \big[2^{-j} m, 2^{-j}(m+1)\big), \quad j \in \mathbb{N}_{-1},\ m \in \mathbb{Z}. \tag{2.125}$$

We return later on in Theorem 2.15 to the (original) homogeneous version of this assertion and to its n-dimensional extension, Remark 2.22. This goes back to J. Marcinkiewicz, [Mar37], who in turn relied on [Pal32]. A recent proof (homogeneous, n-dimensional) using wavelet techniques may be found in [Woj97, Section 8.3]. One may also consult [Woj91, Theorem 13, p. 63] and [LiZ79, Theorem 2.c.5, pp. 155/156].

Remark 2.12. By the discussion in Remark 2.10 the restrictions for p, q, s in part (i) of the above theorem are natural in case of the spaces $B^s_{pq}(\mathbb{R})$. As for the spaces $F^s_{pq}(\mathbb{R})$ it is not so clear to which extent the restrictions in (2.112) caused by q are necessary or natural. Let

$$H^s_p(\mathbb{R}) = F^s_{p,2}(\mathbb{R}), \quad 0 < p < \infty,\ s \in \mathbb{R}, \tag{2.126}$$

be the (fractional Hardy–)Sobolev spaces according to (1.17)–(1.19), extended to $p \le 1$. Then (2.112) reduces to

$$\begin{cases} 2 \le p < \infty, & -\frac{1}{2} < s < \frac{1}{p}, \\ p < 2, & \frac{1}{p} - 1 < s < \frac{1}{2}, \end{cases} \tag{2.127}$$

Figure 2.3. It is not so clear what happens outside the shaded H-region compared with the shaded B-region.

2.2.4 Haar bases on intervals

Let $I = (0,1)$ be the unit interval on $\mathbb{R}$ and let

$$\{h_0, h_{jm} : j \in \mathbb{N}_0;\ m = 0, \dots, 2^j - 1\} \tag{2.128}$$

be the same L_∞-normalised orthogonal Haar basis in $L_2(I)$ as in (2.1), (2.2). Recall that h_0 is the characteristic function of I and that

$$h_{jm}(x) = \begin{cases} 1 & \text{if } 2^{-j}m \le x < 2^{-j}m + 2^{-j-1}, \\ -1 & \text{if } 2^{-j}m + 2^{-j-1} \le x < 2^{-j}(m+1), \\ 0 & \text{otherwise.} \end{cases} \tag{2.129}$$

This coincides essentially with the restriction of the system (2.93)–(2.96) to I (with $h_0 = h_F$ in place of $h_{-1,0}$). We wish to extend the classical assertions for $L_p(I)$, $1 < p < \infty$, collected in Theorem 2.1 (i) to suitable spaces $A^s_{pq}(I)$. Recall that $A^s_{pq}(I)$ is the restriction of $A^s_{pq}(\mathbb{R})$ to I according to Definition 1.24 (i). For this purpose we need the counterpart of the sequence spaces in (2.100)–(2.102). Let $b_{pq}(I)$ and $f_{pq}(I)$ with $0 < p, q \le \infty$ be the spaces of all sequences

$$\mu = \{\mu_0, \mu_{jm} : j \in \mathbb{N}_0;\ m = 0, \dots, 2^j - 1\} \subset \mathbb{C} \tag{2.130}$$

quasi-normed by

$$\|\mu \,| b_{pq}(I)\| = |\mu_0| + \Big(\sum_{j=0}^{\infty} \Big(\sum_{m=0}^{2^j-1} |\mu_{jm}|^p \Big)^{q/p} \Big)^{1/q} \tag{2.131}$$

and

$$\|\mu \,| f_{pq}(I)\| = |\mu_0| + \Big\| \Big(\sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} |\mu_{jm}\, \chi^{(p)}_{jm}(\cdot)|^q \Big)^{1/q} |L_p(I) \Big\|, \tag{2.132}$$

usual modification if $p = \infty$ and/or $q = \infty$. Here $\chi^{(p)}_{jm}(x) = 2^{j/p}\chi_{jm}(x)$, where χ_{jm} is the characteristic function of the interval

$$I_{jm} = \big[2^{-j}m,\ 2^{-j}(m+1)\big), \quad j \in \mathbb{N}_0;\ m = 0, \dots, 2^j - 1. \tag{2.133}$$

Theorem 2.13. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, *and*

$$\frac{1}{p} - 1 < s < \min\Big(1, \frac{1}{p}\Big). \tag{2.134}$$

Let $f \in D'(I)$. *Then* $f \in B^s_{pq}(I)$ *if, and only if, it can be represented as*

$$f = \mu_0 h_0 + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} \mu_{jm}\, 2^{-j(s-\frac{1}{p})} h_{jm}, \quad \mu \in b_{pq}(I), \tag{2.135}$$

unconditional convergence being in $B^{\sigma}_{pq}(I)$ *with* $\sigma < s$. *The representation* (2.135) *is unique,*

$$\begin{cases} \mu_0 = \mu_0(f) = \int_I f(x) h_0(x)\, \mathrm{d}y, \\ \mu_{jm} = \mu_{jm}(f) = 2^{j(s-\frac{1}{p}+1)} \int_I f(x) h_{jm}(x)\, \mathrm{d}x, \; j \in \mathbb{N}_0;\; m = 0, \ldots, 2^j - 1, \end{cases} \tag{2.136}$$

and

$$J : f \mapsto \mu(f) \tag{2.137}$$

is an isomorphic map of $B^s_{pq}(I)$ *onto* $b_{pq}(I)$. *If, in addition,* $p < \infty$, $q < \infty$, *then* (2.128) *is an unconditional basis in* $B^s_{pq}(I)$.

(ii) *Let*

$$\begin{cases} 0 < p < \infty,\; 0 < q < \infty, & \max\left(\frac{1}{p}, \frac{1}{q}, 1\right) - 1 < s < \min\left(\frac{1}{p}, \frac{1}{q}, 1\right), \\ 1 < p < \infty,\; 1 < q < \infty, & s = 0, \\ 1 < p < \infty,\; 1 < q \le \infty, & \max\left(\frac{1}{p}, \frac{1}{q}\right) - 1 < s < 0. \end{cases} \tag{2.138}$$

Let $f \in D'(I)$. *Then* $f \in F^s_{pq}(I)$ *if, and only if, it can be represented as*

$$f = \mu_0 h_0 + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} \mu_{jm} 2^{-j(s-\frac{1}{p})} h_{jm}, \quad \mu \in f_{pq}(I), \tag{2.139}$$

unconditional convergence being in $F^{\sigma}_{pq}(I)$ *with* $\sigma < s$. *The representation* (2.139) *is unique with* $\mu_0(f)$ *and* $\mu_{jm}(f)$ *as in* (2.136). *Furthermore,* J *in* (2.137) *is an isomorphic map of* $F^s_{pq}(I)$ *onto* $f_{pq}(I)$. *If, in addition,* $q < \infty$ *then* (2.128) *is an unconditional basis in* $F^s_{pq}(I)$.

Proof. Recall that the characteristic function $\chi = h_0$ of I is a pointwise multiplier for

$$A^s_{pq}(\mathbb{R}), \quad s \in \mathbb{R},\; 0 < p \le \infty,\; 0 < q \le \infty, \tag{2.140}$$

($p < \infty$ for the F-spaces) if, and only if, $\frac{1}{p} - 1 < s < \frac{1}{p}$. One may consult [RuS96, Section 4.6.3, pp. 208, 258] for a proof and references. In particular, all spaces of interest can be identified with

$$\widetilde{A}^s_{pq}\big([0,1]\big) = \big\{g \in A^s_{pq}(\mathbb{R}) : \operatorname{supp} g \subset [0,1]\big\} \tag{2.141}$$

in the notation of Definition 1.24, Remark 1.25. Then the above theorem follows from Theorem 2.9. □

Remark 2.14. We complement now the references given in Remark 2.10. Recall that

$$L_p(I) = F^0_{p,2}(I), \quad 1 < p < \infty, \tag{2.142}$$

as the restriction of (2.120). Then it follows that the Haar system (2.128) is an unconditional basis in $L_p(I)$ with $1 < p < \infty$ and that one has an obvious counterpart of the

Littlewood–Paley assertion (2.124). This is a new proof of the corresponding classical assertion in Theorem 2.1 (i). One may also consult the corresponding references at the end of the proof (Step 5) of this theorem and in Remark 2.11. Furthermore we mention the elaborated theory of spline systems on intervals due to Z. Ciesielski, [Cie75], covering in particular Haar systems. This had been used in [Rop76] to prove the above assertion for the spaces $B^s_{pq}(I)$ with $1 \le p, q \le \infty$, $0 < s < \frac{1}{p}$. A more recent proof based on wavelet techniques may be found in [KaL95, Chapter 6, Section 5, pp. 328–335]. As for further information and references about spline bases in function spaces we refer to [Tri81], [T83, Section 2.12.3] and [Kam96]. In Section 2.5 below we return to spline bases.

2.2.5 Littlewood–Paley theorem

So far we have the Littlewood–Paley characterisation (2.122)–(2.124) of the spaces $L_p(\mathbb{R})$ with $1 < p < \infty$ in terms of the inhomogeneous Haar system (2.121). But this can be used rather easily to prove the corresponding classical assertion based on homogeneous Haar bases. Let $h_M(x)$ be the same function as in (2.93). Then the homogeneous Haar system

$$\{h^{jm} : j \in \mathbb{Z},\ m \in \mathbb{Z}\} \tag{2.143}$$

with

$$h^{jm}(x) = h_M(2^j x - m), \quad j \in \mathbb{Z},\ m \in \mathbb{Z}, \tag{2.144}$$

is the counterpart of the (inhomogeneous) Haar system (2.95), (2.96). If $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}$ then $h^{jm} = h_{jm}$. It is orthogonal in $L_2(\mathbb{R})$. One can replace χ_{jm} in (2.124) by h_{jm}. Then one has the following homogeneous counterpart.

Theorem 2.15. *Let $1 < p < \infty$. The homogeneous Haar system (2.143) is an unconditional basis in $L_p(\mathbb{R})$. Furthermore $f \in L_p(\mathbb{R})$ can be expanded as*

$$f = \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \lambda^{jm} h^{jm} \tag{2.145}$$

with

$$\lambda^{jm} = 2^j \int_{\mathbb{R}} f(x)\, h^{jm}(x)\, \mathrm{d}x, \quad j \in \mathbb{Z},\ m \in \mathbb{Z}, \tag{2.146}$$

and

$$\| f \,|L_p(\mathbb{R})\| \sim \Big\| \Big(\sum_{j,m} |\lambda^{jm} h^{jm}(\cdot)|^2 \Big)^{1/2} |L_p(\mathbb{R}) \Big\| \tag{2.147}$$

(*equivalent norms*).

Proof. It is sufficient to prove the theorem for functions $f \in L_p(\mathbb{R})$ with $f(x) = 0$ if $x < 0$. Let $\mathbb{R}_+ = (0, \infty)$,

$$\varphi \in D(\mathbb{R}_+), \quad 0 \le \varphi(x) \le 1, \quad \int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = 1, \tag{2.148}$$

and $\varphi_\varepsilon(x) = \varepsilon\,\varphi(\varepsilon x)$, $\varepsilon > 0$. Assume in addition that f has a compact support. Let

$$f_\varepsilon(x) = f(x) - \int_{\mathbb{R}} f(y)\,\mathrm{d}y\;\varphi_\varepsilon(x). \tag{2.149}$$

Then

$$\int_{\mathbb{R}} f_\varepsilon(x)\,\mathrm{d}x = 0 \quad \text{and} \quad \|f - f_\varepsilon \,|L_p(\mathbb{R})\| \le \varepsilon^{1-\frac{1}{p}}\, \|f\,|L_1(\mathbb{R})\| \to 0 \tag{2.150}$$

if $\varepsilon \to 0$. Hence the functions $g \in L_p(\mathbb{R})$ with compact support in $\mathbb{R}_+$ and $\int_{\mathbb{R}} g(y)\,\mathrm{d}y = 0$ are dense in $L_p(\mathbb{R}_+)$. If g is a function of this type then for some $k \in \mathbb{N}$,

$$\operatorname{supp} g(2^k\cdot) \subset I = (0,1), \quad \int_I g(2^k y)\,\mathrm{d}y = 0. \tag{2.151}$$

We expand $g(2^k x)$ according to (2.122). There is no term with $j = -1$, hence

$$g\big(2^k x\big) = \sum_{j=0}^{\infty} \sum_{l \in \mathbb{Z}} \lambda_{jl}\, h_M\big(2^j x - l\big), \tag{2.152}$$

and

$$g(x) = \sum_{j=0}^{\infty} \sum_{l \in \mathbb{Z}} \lambda_{jl}\, h_M\big(2^{j-k} x - l\big) = \sum_{j=0}^{\infty} \sum_{l \in \mathbb{Z}} \lambda_{jl}\, h^{j-k,l}(x). \tag{2.153}$$

The rest is now a matter of orthogonality, completion, the (inhomogeneous) equivalence (2.124), and the homogeneity

$$\|f(\lambda x)\,|L_p(\mathbb{R})\| = \lambda^{-1/p}\, \|f\,|L_p(\mathbb{R})\|, \quad \lambda > 0, \; f \in L_p(\mathbb{R}). \tag{2.154}$$

□

Remark 2.16. This is the standard version how the Littlewood–Paley theorem for L_p-spaces with $1 < p < \infty$ in terms of Haar systems is presented in the literature, including the references given in Remark 2.11.

2.3 Haar wavelet bases on $\mathbb{R}^n$ and on cubes

2.3.1 Inequalities

We wish to extend the assertions in the previous Section 2.2 about Haar bases in function spaces on $\mathbb{R}$ and on intervals to higher dimensions. First we ask for counterparts of the crucial inequalities in Propositions 2.6, 2.8 in higher dimensions. We use the notation introduced in Section 1.1.2. Let again Q_{jm} be cubes in $\mathbb{R}^n$, $n \in \mathbb{N}$, with sides parallel to the axes of coordinates centred at $2^{-j}m$ with side-length 2^{-j} where $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$. Let χ_{jm} be the characteristic function of Q_{jm} and let

$$\chi_{jm}^{(p)}(x) = 2^{jn/p}\, \chi_{jm}(x), \quad x \in \mathbb{R}^n,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{2.155}$$

be its p-normalised modification where $0 < p \le \infty$. Let b_{pq} and f_{pq} be the n-dimensional sequence spaces introduced in Definition 1.3 where again $n \in \mathbb{N}$ will not be indicated. The convergence of the series below must be understood as in Theorem 1.7 and as in the references mentioned in Remark 1.8. We will not stress this point in the sequel.

Proposition 2.17. *Let $n \in \mathbb{N}$ and let b_{pq} and f_{pq} be the n-dimensional sequence spaces according to Definition 1.3.*

(i) *Let $0 < p,q \le \infty$ and*

$$n\Big(\max\Big(\frac{1}{p},1\Big) - 1\Big) < s < \frac{1}{p}. \tag{2.156}$$

Then

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \mu_{jm}\, 2^{-j(s-\frac{n}{p})}\, \chi_{jm}, \quad \mu \in b_{pq}, \tag{2.157}$$

belongs to $B^s_{pq}(\mathbb{R}^n)$ and

$$\| f \,|B^s_{pq}(\mathbb{R}^n)\| \le c\, \|\mu \,|b_{pq}\| \tag{2.158}$$

for some $c > 0$ and all $\mu \in b_{pq}$.

(ii) *Let $0 < p,q < \infty$ and*

$$n\Big(\max\Big(\frac{1}{p},\frac{1}{q},1\Big) - 1\Big) < s < \min\Big(\frac{1}{p},\frac{1}{q}\Big). \tag{2.159}$$

Then

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \mu_{jm}\, 2^{-j(s-\frac{n}{p})}\, \chi_{jm}, \quad \mu \in f_{pq}, \tag{2.160}$$

belongs to $F^s_{pq}(\mathbb{R}^n)$ and

$$\| f \,|F^s_{pq}(\mathbb{R}^n)\| \le c\, \|\mu \,|f_{pq}\| \tag{2.161}$$

for some $c > 0$ and all $\mu \in f_{pq}$.

Proof. Step 1. We prove part (ii). The case $n = 1$ is covered by Proposition 2.6 (ii). Let $n \geq 2$ and let f be given by (2.160),

$$f = \sum_{j=0}^{\infty} \sum_{m_1 \in \mathbb{Z}} \mu_{j,m_1}(x')\, 2^{-j(s-\frac{1}{p})}\, \chi_{j,m_1}(x_1) \tag{2.162}$$

with

$$\mu_{j,m_1}(x') = \sum_{m' \in \mathbb{Z}^{n-1}} \mu_{j,(m_1,m')}\, 2^{j\frac{n-1}{p}}\, \chi_{j,m'}(x') \tag{2.163}$$

where $x = (x_1, x') \in \mathbb{R}^n$, $x' \in \mathbb{R}^{n-1}$. Then it follows from Proposition 2.6 (ii) that

$$\begin{aligned} \|f(\cdot, x')\,|F^s_{pq}(\mathbb{R})\| &\leq c \Big\| \Big(\sum_{j=0}^{\infty} \sum_{m_1 \in \mathbb{Z}} |\mu_{j,m_1}(x')\, 2^{j/p}\, \chi_{j,m_1}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}) \Big\| \\ &\leq c' \Big\| \Big(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\mu_{jm}\, 2^{jn/p}\, \chi_{jm}(\cdot, x')|^q \Big)^{1/q} |L_p(\mathbb{R}) \Big\|. \end{aligned} \tag{2.164}$$

Let $f(x)$ be a measurable function on $\mathbb{R}^n$ and let

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad x^l = (x_1, .., x_{l-1}, x_{l+1}, .., x_n) \in \mathbb{R}^{n-1}, \tag{2.165}$$

where $l = 1, \ldots, n$. Then f^{x^l},

$$x_l \mapsto f^{x^l}(x_l) = f(x), \quad x \in \mathbb{R}^n, \tag{2.166}$$

is considered as a function on $\mathbb{R}$ for any fixed $x^l \in \mathbb{R}^{n-1}$. The left-hand side of (2.159) ensures that $F^s_{pq}(\mathbb{R}^n)$ has the Fubini property

$$\|f\,|F^s_{pq}(\mathbb{R}^n)\| \sim \sum_{l=1}^{n} \big\| \, \|f^{x^l}\,|F^s_{pq}(\mathbb{R})\| \,\big|L_p(\mathbb{R}^{n-1}) \big\|, \tag{2.167}$$

[T01, Theorem 4.4, p. 36]. Then (2.161) follows from (2.164), (2.167).

Step 2. We prove part (i). The assumption (2.156) excludes $p = \infty$. Recall $F^s_{pp}(\mathbb{R}^n) = B^s_{pp}(\mathbb{R}^n)$, $p < \infty$, similarly $b_{pp} = f_{pp}$. By Theorem 1.22 one has the real interpolation

$$\big(B^{s_0}_{pp}(\mathbb{R}^n), B^{s_1}_{pp}(\mathbb{R}^n)\big)_{\theta,q} = B^s_{pq}(\mathbb{R}^n) \tag{2.168}$$

with $0 < \theta < 1$ and

$$0 < p < \infty, \quad 0 < q \leq \infty, \quad -\infty < s_0 < s_1 < \infty, \quad s = (1-\theta)s_0 + \theta s_1. \tag{2.169}$$

One can reformulate the assertions of the proposition in terms of $\lambda = \{\lambda_{jm}\}$ with $\lambda_{jm} = 2^{-j(s-\frac{n}{p})}\mu_{jm}$. Then one has to replace b_{pq} and f_{pq} by $\bar{b}^s_{pq}$ and $\bar{f}^s_{pq}$ according to Definition 1.11. The sequence counterpart of (2.168) is given by

$$\big(\bar{b}^{s_0}_{pp}, \bar{b}^{s_1}_{pp}\big)_{\theta,q} = \bar{b}^s_{pq}. \tag{2.170}$$

This follows from the isomorphic map in Theorem 1.18 (restricted to one G-component). But the assertion itself is much simpler and can be proved directly as in [T78, Theorem 1.18.2, pp. 123–125] (extended to quasi-Banach spaces). Then part (i) of the proposition follows from part (ii) by real interpolation. □

We need the n-dimensional counterpart of Proposition 2.8. For this purpose we modify the construction of wavelets in $\mathbb{R}^n$ according to (1.55)–(1.59) as follows. Instead of (1.55), (1.56) we have now

$$h_F \in L_\infty(\mathbb{R}), \quad h_M \in L_\infty(\mathbb{R}), \tag{2.171}$$

with h_F, h_M as in (2.93), (2.94) and

$$\int_{\mathbb{R}} h_M(x)\,\mathrm{d}x = 0. \tag{2.172}$$

Let

$$G = (G_1, \dots, G_n) \in G^0 = \{F, M\}^n \tag{2.173}$$

and

$$G = (G_1, \dots, G_n) \in G^j = \{F, M\}^{n*}, \quad j \in \mathbb{N}, \tag{2.174}$$

be as in (1.57), (1.58). Then

$$\big\{h^G_{jm} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n\big\} \tag{2.175}$$

with

$$h^G_{jm}(x) = \prod_{r=1}^n h_{G_r}\big(2^j x_r - m_r\big), \quad j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n, \tag{2.176}$$

is the orthogonal *Haar wavelet basis* in $L_2(\mathbb{R}^n)$ we are looking for. We prefer now an L_∞-normalisation as in (2.95), (2.96) instead of the L_2-normalisation in (1.59). First we complement the sequence spaces in Definition 1.3 by an additional summation over G^j according to (2.173), (2.174). Recall that $\chi^{(p)}_{jm}$ is the p-normalised characteristic function χ_{jm} of the cube Q_{jm} as introduced in (2.155).

Definition 2.18. Let $n \in \mathbb{N}$, $0 < p \le \infty$, $0 < q \le \infty$. Then $b_{pq}(\mathbb{R}^n)$ is the collection of all sequences

$$\lambda = \big\{\lambda^G_{jm} \in \mathbb{C} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n\big\} \tag{2.177}$$

such that

$$\|\lambda\,|b_{pq}(\mathbb{R}^n)\| = \Big(\sum_{j=0}^\infty \sum_{G\in G^j} \Big(\sum_{m\in\mathbb{Z}^n} |\lambda^G_{jm}|^p\Big)^{q/p}\Big)^{1/q} < \infty \tag{2.178}$$

and $f_{pq}(\mathbb{R}^n)$ is the collection of all sequences (2.177) such that

$$\|\lambda\,|f_{pq}(\mathbb{R}^n)\| = \Big\|\Big(\sum_{j,G,m} |\lambda^G_{jm}\,\chi^{(p)}_{jm}(\cdot)|^q\Big)^{1/q} |L_p(\mathbb{R}^n)\Big\| < \infty \tag{2.179}$$

with the usual modifications if $p = \infty$ and/or $q = \infty$.

Remark 2.19. One has $b_{pp}(\mathbb{R}^n) = f_{pp}(\mathbb{R}^n)$. Otherwise $b_{pq}(\mathbb{R}^n)$ and $f_{pq}(\mathbb{R}^n)$ are the n-dimensional versions of the sequence spaces in (2.100)–(2.102). We prefer now a different normalisation of the spaces in Definition 1.17.

We need the n-dimensional version of the local means in (2.97), now with respect to the system (2.175),

$$k^G_{jm}(f) = \int_{\mathbb{R}^n} k^G_{jm}(y)\, f(y)\, \mathrm{d}y, \quad k^G_{jm}(y) = 2^{jn} h^G_{jm}(y), \tag{2.180}$$

where $j \in \mathbb{N}_0$, $G \in G^j$, and $m \in \mathbb{Z}^n$. For $s \in \mathbb{R}$ and $0 < p \le \infty$ one has (at least in $L_2(\mathbb{R}^n)$)

$$f = \sum_{j,G,m} k^G_{jm}(f)\, h^G_{jm} = \sum_{j,G,m} \mu^G_{jm}(f)\, 2^{-j(s-\frac{n}{p})}\, h^G_{jm} \tag{2.181}$$

with

$$\mu^G_{jm}(f) = 2^{j(s-\frac{n}{p})}\, k^G_{jm}(f), \quad j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n. \tag{2.182}$$

This is the same normalisation as in (2.157), (2.160). The sequence spaces $b_{pq}(\mathbb{R}^n)$ and $f_{pq}(\mathbb{R}^n)$ have the same meaning as in Definition 2.18. Let $\mu(f)$ be the sequence $\lambda = \mu(f)$ according to (2.177) with $\lambda^G_{jm} = \mu^G_{jm}(f)$.

Proposition 2.20. *Let* $n \in \mathbb{N}$.

(i) *Let* $0 < p, q \le \infty$ *and*

$$n\left(\max\left(\frac{1}{p}, 1\right) - 1\right) < s < 1. \tag{2.183}$$

Then there is a $c > 0$ *such that*

$$\|\mu(f)\,|b_{pq}(\mathbb{R}^n)\| \le c\, \|f\,|B^s_{pq}(\mathbb{R}^n)\| \quad \textit{for all } f \in B^s_{pq}(\mathbb{R}^n). \tag{2.184}$$

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$ *and*

$$n\left(\max\left(\frac{1}{p}, \frac{1}{q}, 1\right) - 1\right) < s < 1. \tag{2.185}$$

Then there is a $c > 0$ *such that*

$$\|\mu(f)\,|f_{pq}(\mathbb{R}^n)\| \le c\, \|f\,|F^s_{pq}(\mathbb{R}^n)\| \quad \textit{for all } f \in F^s_{pq}(\mathbb{R}^n). \tag{2.186}$$

Proof. The kernels k^G_{jm} in (2.180) satisfy the conditions for k_{jm} in Definition 1.9 with $A = 0$ and $B = 1$. Then the above proposition follows from Theorem 1.15 with σ_p and σ_{pq} as in (1.35) and the normalisation (2.182). □

2.3.2 Haar wavelet bases on $\mathbb{R}^n$

The proof of Theorem 2.9 about Haar bases on $\mathbb{R}$ relied on the preceding Propositions 2.6, 2.8. The Propositions 2.17, 2.20 are their n-dimensional generalisations. The other arguments used in the proof of Theorem 2.9 such as duality and interpolation have direct n-dimensional counterparts. Let $b_{pq}(\mathbb{R}^n)$ and $f_{pq}(\mathbb{R}^n)$ be the sequence spaces according to Definition 2.18. The Haar system $\{h^G_{jm}\}$ has the same meaning as in (2.175), (2.176). We use $\mu(f)$ as explained after (2.182).

Theorem 2.21. *Let* $n \in \mathbb{N}$.
(i) *Let* $0 < p \le \infty$, $0 < q \le \infty$, *and*

$$\max\Big(n\Big(\frac{1}{p}-1\Big), \frac{1}{p}-1\Big) < s < \min\Big(\frac{1}{p}, 1\Big). \tag{2.187}$$

Let $f \in S'(\mathbb{R}^n)$. *Then* $f \in B^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \mu^G_{jm}\, 2^{-j(s-\frac{n}{p})}\, h^G_{jm}, \quad \mu \in b_{pq}(\mathbb{R}^n), \tag{2.188}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ *and locally in any space* $B^\sigma_{pq}(\mathbb{R}^n)$ *with* $\sigma < s$. *The representation* (2.188) *is unique,*

$$\mu^G_{jm} = \mu^G_{jm}(f) = 2^{j(s-\frac{n}{p}+n)} \int_{\mathbb{R}^n} f(x)\, h^G_{jm}(x)\, \mathrm{d}x, \quad j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n, \tag{2.189}$$

and

$$J:\ f \mapsto \mu(f) \tag{2.190}$$

is an isomorphic map of $B^s_{pq}(\mathbb{R}^n)$ *onto* $b_{pq}(\mathbb{R}^n)$. *If, in addition,* $p < \infty$, $q < \infty$, *then*

$$\big\{2^{-j(s-\frac{n}{p})} h^G_{jm} :\ j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n\big\} \tag{2.191}$$

is an unconditional (*normalised*) *basis in* $B^s_{pq}(\mathbb{R}^n)$.

(ii) *Let*

$$\begin{cases} 0 < p < \infty, 0 < q < \infty, & n\big(\max\big(\frac{1}{p}, \frac{1}{q}, 1\big) - 1\big) < s < \min\big(\frac{1}{p}, \frac{1}{q}, 1\big), \\ 1 < p < \infty, 1 < q < \infty, & s = 0, \\ 1 < p < \infty, 1 < q \le \infty, & \max\big(\frac{1}{p}, \frac{1}{q}\big) - 1 < s < 0. \end{cases} \tag{2.192}$$

Let $f \in S'(\mathbb{R}^n)$. *Then* $f \in F^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \mu^G_{jm}\, 2^{-j(s-\frac{n}{p})}\, h^G_{jm}, \quad \mu \in f_{pq}(\mathbb{R}^n), \tag{2.193}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ and locally in any space $F^\sigma_{pq}(\mathbb{R}^n)$ with $\sigma < s$. The representation (2.193) *is unique with $\mu^G_{jm}(f)$ as in* (2.189). *Furthermore, J in* (2.190) *is an isomorphic map of $F^s_{pq}(\mathbb{R}^n)$ onto $f_{pq}(\mathbb{R}^n)$. If, in addition, $q < \infty$ then* (2.191) *is an unconditional (normalised) basis in $F^s_{pq}(\mathbb{R}^n)$.*

Proof. We are now in the same position as in the proof of Theorem 2.9. The comments and the references about the technicalities are the same as there in Step 1. Instead of Propositions 2.6, 2.8 we rely now in the counterpart of Step 2 on Propositions 2.17, 2.20 and the representation (2.181). Step 3 is based on the duality (1.77), now n-dimensional, and the interpolation (2.117), where one can replace $\mathbb{R}$ by $\mathbb{R}^n$. Similarly one can extend the arguments in Step 4 from $\mathbb{R}$ to $\mathbb{R}^n$. □

Remark 2.22. The comments and references in Remark 2.10 apply also to the n-dimensional case. In Remark 2.11 we discussed the classical Littlewood–Paley characterisation of $L_p(\mathbb{R})$, $1 < p < \infty$, in terms of Haar bases. Now one obtains as a special case of the above theorem a corresponding *Littlewood–Paley characterisation of $L_p(\mathbb{R}^n)$, $1 < p < \infty$, in terms of the Haar system*

$$\big\{h^G_{jm} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n\big\}. \tag{2.194}$$

In particular, $\{h^G_{jm}\}$ is an unconditional basis in $L_p(\mathbb{R}^n)$, $1 < p < \infty$,

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \lambda^G_{jm}\, h^G_{jm} \tag{2.195}$$

with

$$\lambda^G_{jm} = 2^{jn} \int_{\mathbb{R}^n} f(x)\, h^G_{jm}(x)\, \mathrm{d}x, \quad j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n, \tag{2.196}$$

and

$$\|f \,|L_p(\mathbb{R}^n)\| \sim \Big\| \Big(\sum_{j,G,m} |\lambda^G_{jm}\, \chi_{jm}(\cdot)|^2 \Big)^{1/2} |L_p(\mathbb{R}^n) \Big\| \tag{2.197}$$

where again χ_{jm} is the characteristic function of the cube Q_{jm} in $\mathbb{R}^n$ centred at $2^{-j}m$ and with side-length 2^{-j}. The homogeneous version of the Littlewood–Paley characterisation for $L_p(\mathbb{R})$, $1 < p < \infty$, according to Theorem 2.15 can also be extended from $\mathbb{R}$ to $\mathbb{R}^n$ by the same arguments as in the proof of this theorem. References and historical comments may be found in Remark 2.11.

For the B^s_{pq}-spaces (with $p < \infty$, $q < \infty$) one has Haar wavelet bases if $\big(\frac{1}{p}, s\big)$ belongs to the shaded region in Figure 2.4. The situation for the F^s_{pq}-spaces is more complicated. We complement now the one-dimensional case for the (fractional Hardy–) Sobolev spaces as discussed in (2.126), (2.127), Figure 2.3, p. 82, by its n-dimensional counterpart

$$H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n), \quad 0 < p < \infty,\ s \in \mathbb{R}. \tag{2.198}$$

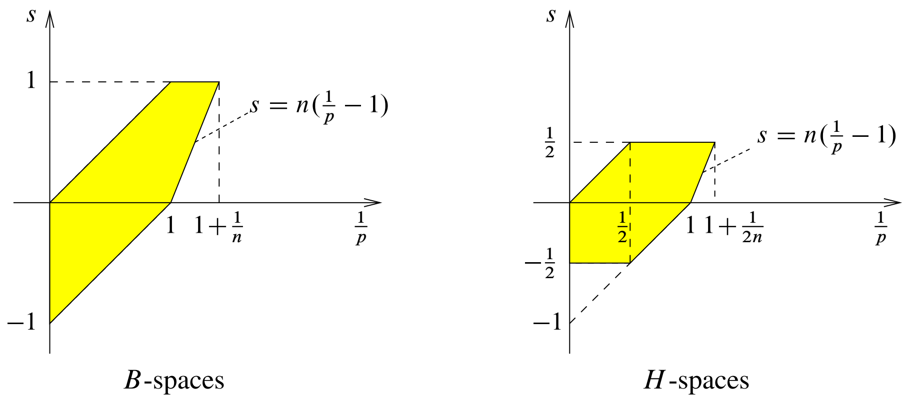

Figure 2.4. Haar bases, $n \in \mathbb{N}$.

Corollary 2.23. *Let $n \in \mathbb{N}$ and let*

$$\begin{cases} 2 \le p < \infty, & -\frac{1}{2} < s < \frac{1}{p}, \\ 1 \le p < 2, & \frac{1}{p} - 1 < s < \frac{1}{2}, \\ p < 1, & n\big(\frac{1}{p} - 1\big) < s < \frac{1}{2}, \end{cases} \tag{2.199}$$

Figure 2.4. Then $\{2^{-j(s-\frac{n}{p})} h^G_{jm}\}$ according to (2.191) is an unconditional (normalised) basis in $H^s_p(\mathbb{R}^n)$ and

$$\|f \,|H^s_p(\mathbb{R}^n)\| \sim \Big\| \Big(\sum_{j,G,m} |\mu^G_{jm}(f)\, \chi^{(p)}_{jm}(\cdot)|^2 \Big)^{1/2} |L_p(\mathbb{R}^n) \Big\| \tag{2.200}$$

with $\mu^G_{jm}(f)$ as in (2.189), (2.193).

Proof. This is a special case of Theorem 2.21 (ii) with $q = 2$. □

For sake of completeness we take over the following assertion from [T06, Theorem 1.58 (iii)].

Proposition 2.24. *The Haar system (2.191) is not a basis in $B^s_{pq}(\mathbb{R}^n)$ if*

$$\begin{cases} \text{either } 1 < p < \infty, & s \notin \big[\frac{1}{p} - 1, \frac{1}{p}\big],\ 0 < q < \infty, \\ \text{or } \frac{n}{n+1} \le p \le 1, & s \notin \big[n\big(\frac{1}{p} - 1\big), 1\big],\ 0 < q < \infty, \\ \text{or } 0 < p < \frac{n}{n+1}, & s \in \mathbb{R},\ 0 < q < \infty. \end{cases} \tag{2.201}$$

Remark 2.25. Details and references may be found in [T06, pp. 29/30]. The assertion itself goes back to [Tri78]. In other words, outside the shaded region for the B-spaces in Figure 2.4, p. 94, the Haar system (2.191) is not a basis in $B^s_{pq}(\mathbb{R}^n)$. This shows that part (i) of the above theorem gives a final answer about Haar wavelet bases in B-spaces with exception of borderline cases. In case of H-spaces, and even more of F-spaces the situation is not so clear. One can expect affirmative answers about Haar bases for H-spaces or F-spaces in some $\big(\frac{1}{p}, s\big)$-regions only if this applies also for B-spaces. This follows from real interpolation. But it is unclear what happens, say, outside the shaded region for H-spaces, but inside the shaded region for B-spaces in Figure 2.4. In case of B-spaces there are at least a few assertions in literature about Haar bases mentioned in Remarks 2.10, 2.14, especially [Cie75], [Rop76], [KaL95], [Tri73c], [Tri78], the relevant parts in [T78] and the above references to [T06]. But we could not find in literature corresponding assertions for, say, H^s_p-spaces, beyond L_p. Something can be said about spaces $F^s_{pq}(\mathbb{R}^n)$ which can be reached by complex interpolation of $B^{s_1}_{p_1 p_1}(\mathbb{R}^n) = F^{s_1}_{p_1 p_1}(\mathbb{R}^n)$ covered by part (i) of the above theorem and $L_{p_2}(\mathbb{R}^n) = F^0_{p_2,2}(\mathbb{R}^n)$, $1 < p_2 < \infty$, [Tri81]. But this is very selective, excluding, for example, spaces $H^s_p(\mathbb{R}^n)$ with $s \neq 0$, $p \neq 2$.

2.3.3 Haar wavelet bases on cubes

Let, as before in (1.230),

$$\mathbb{Q}^n = \big\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : 0 < x_l < 1\big\} \tag{2.202}$$

be the unit cube in $\mathbb{R}^n$, $2 \le n \in \mathbb{N}$. Let

$$\mathbb{P}^n_j = \big\{m \in \mathbb{Z}^n : 0 \le m_l \le 2^j - 1;\ l = 1, \dots, n\big\}, \quad j \in \mathbb{N}_0. \tag{2.203}$$

Then

$$\big\{h^G_{jm} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{P}^n_j\big\} \tag{2.204}$$

is the restriction of (2.175), (2.176) to $\mathbb{Q}^n$. We need the n-dimensional version of the sequence spaces $b_{pq}(I)$ and $f_{pq}(I)$ on the interval I according to (2.130)–(2.132). Let

$$\lambda = \big\{\lambda^G_{jm} \in \mathbb{C} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{P}^n_j\big\} \tag{2.205}$$

be the restriction of (2.177) to $\mathbb{Q}^n$. Let $0 < p, q \le \infty$. Then $b_{pq}(\mathbb{Q}^n)$ is the collection of all sequences (2.205) such that

$$\|\lambda \,|b_{pq}(\mathbb{Q}^n)\| = \Big(\sum_{j=0}^\infty \sum_{G \in G^j} \Big(\sum_{m \in \mathbb{P}^n_j} |\lambda^G_{jm}|^p\Big)^{q/p}\Big)^{1/q} < \infty, \tag{2.206}$$

and $f_{pq}(\mathbb{Q}^n)$ is the collection of all sequences (2.205) such that

$$\|\lambda \,|f_{pq}(\mathbb{Q}^n)\| = \Big\|\Big(\sum_{j,G,m} |\lambda^G_{jm}\, \chi^{(p)}_{jm}(\cdot)|^q\Big)^{1/q} |L_p(\mathbb{Q}^n)\Big\| < \infty, \tag{2.207}$$

with the usual modifications if $p = \infty$ and/or $q = \infty$. This is the restriction of $b_{pq}(\mathbb{R}^n)$ and $f_{pq}(\mathbb{R}^n)$ according to Definition 2.18 to $\mathbb{Q}^n$, where $\chi_{jm}^{(p)}$ has the same meaning as there. One can replace $\chi_{jm}^{(p)}$ in (2.207) by $h_{jm}^{G,(p)} = 2^{jn/p} h_{jm}^G$ and/or $L_p(\mathbb{Q}^n)$ by $L_p(\mathbb{R}^n)$. For useful modifications of this type we refer to [T06], Section 1.5.3, pp. 18/19. According to Definition 1.24 the spaces $B^s_{pq}(\mathbb{Q}^n)$ and $F^s_{pq}(\mathbb{Q}^n)$ are restrictions of corresponding spaces on $\mathbb{R}^n$ to $\Omega = \mathbb{Q}^n$.

Theorem 2.26. *Let* $2 \le n \in \mathbb{N}$.

(i) *Let* $0 < p \le \infty$, $0 < q \le \infty$ *and*

$$\max\Big(n\Big(\frac{1}{p} - 1\Big), \frac{1}{p} - 1\Big) < s < \min\Big(\frac{1}{p}, 1\Big), \tag{2.208}$$

Figure 2.4, p. 94. Let $f \in D'(\mathbb{Q}^n)$. *Then* $f \in B^s_{pq}(\mathbb{Q}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{P}^n_j} \mu^G_{jm}\, 2^{-j(s-\frac{n}{p})}\, h^G_{jm}, \quad \mu \in b_{pq}(\mathbb{Q}^n), \tag{2.209}$$

unconditional convergence being in $B^\sigma_{pq}(\mathbb{Q}^n)$ *with* $\sigma < s$. *The representation* (2.209) *is unique,*

$$\mu^G_{jm} = \mu^G_{jm}(f) = 2^{j(s-\frac{n}{p}+n)} \int_{\mathbb{Q}^n} f(x)\, h^G_{jm}(x)\, \mathrm{d}x, \quad j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{P}^n_j, \tag{2.210}$$

and

$$J : f \mapsto \mu(f) \tag{2.211}$$

is an isomorphic map of $B^s_{pq}(\mathbb{Q}^n)$ *onto* $b_{pq}(\mathbb{Q}^n)$. *If, in addition,* $p < \infty$, $q < \infty$, *then*

$$\big\{2^{-j(s-\frac{n}{p})} h^G_{jm} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{P}^n_j\big\} \tag{2.212}$$

is an unconditional (normalised) basis in $B^s_{pq}(\mathbb{Q}^n)$.

(ii) *Let*

$$\begin{cases} 0 < p < \infty,\ 0 < q < \infty, & n\big(\max\big(\frac{1}{p}, \frac{1}{q}, 1\big) - 1\big) < s < \min\big(\frac{1}{p}, \frac{1}{q}, 1\big), \\ 1 < p < \infty,\ 1 < q < \infty, & s = 0, \\ 1 < p < \infty,\ 1 < q \le \infty, & \max\big(\frac{1}{p}, \frac{1}{q}\big) - 1 < s < 0. \end{cases} \tag{2.213}$$

Let $f \in D'(\mathbb{Q}^n)$. *Then* $f \in F^s_{pq}(\mathbb{Q}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \mathbb{P}^n_j} \mu^G_{jm}\, 2^{-j(s-\frac{n}{p})}\, h^G_{jm}, \quad \mu \in f_{pq}(\mathbb{Q}^n), \tag{2.214}$$

unconditional convergence being in $F^\sigma_{pq}(\mathbb{Q}^n)$ *with* $\sigma < s$. *The representation* (2.214) *is unique,* $\mu^G_{jm} = \mu^G_{jm}(f)$, *as in* (2.210). *Furthermore,* J *in* (2.211) *is an isomorphic map of* $F^s_{pq}(\mathbb{Q}^n)$ *onto* $f_{pq}(\mathbb{Q}^n)$. *If, in addition,* $q < \infty$ *then* (2.212) *is an unconditional (normalised) basis in* $F^s_{pq}(\mathbb{Q}^n)$.

Proof. The characteristic function of $\mathbb{Q}^n$ is a pointwise multiplier for

$$A^s_{pq}(\mathbb{R}^n), \quad s \in \mathbb{R},\ 0 < p \le \infty,\ 0 < q \le \infty, \tag{2.215}$$

($p < \infty$ for F-spaces) if, and only if,

$$\max\Big(n\Big(\frac{1}{p} - 1\Big), \frac{1}{p} - 1\Big) < s < \frac{1}{p}. \tag{2.216}$$

We refer again to [RuS96, Section 4.6.3, pp. 208, 258]. Now one obtains the above assertion from Theorem 2.21 in the same way as in the proof of Theorem 2.13. □

Remark 2.27. One does not really need the deep assertion that the characteristic function of $\mathbb{Q}^n$ is a pointwise multiplier in the spaces (2.215), (2.216). It comes out as a by-product for those spaces covered by the above theorem by the same type of reasoning as in the proof of Theorem 2.41 below. But it illuminates the situation. Otherwise the above assertion is the extension of Theorem 2.13 from intervals to cubes. The references in Remarks 2.14 and 2.25 apply partly also to the spaces $B^s_{pq}(\mathbb{Q}^n)$ and $F^s_{pq}(\mathbb{Q}^n)$ on cubes. The restriction of (2.198) to $\mathbb{Q}^n$,

$$H^s_p(\mathbb{Q}^n) = F^s_{p,2}(\mathbb{Q}^n), \quad 0 < p < \infty,\ s \in \mathbb{R}, \tag{2.217}$$

are the (fractional Hardy–)Sobolev spaces on cubes.

Corollary 2.28. *Let* $2 \le n \in \mathbb{N}$ *and let*

$$\begin{cases} 2 \le p < \infty, & -\frac{1}{2} < s < \frac{1}{p}, \\ 1 \le p < 2, & \frac{1}{p} - 1 < s < \frac{1}{2}, \\ p < 1, & n\big(\frac{1}{p} - 1\big) < s < \frac{1}{2}, \end{cases} \tag{2.218}$$

Figure 2.4, p. 94. Then

$$\big\{2^{-j(s-\frac{n}{p})}\, h^G_{jm} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{P}^n_j\big\} \tag{2.219}$$

is an unconditional (normalised) basis in $H^s_p(\mathbb{Q}^n)$,

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{P}^n_j} \mu^G_{jm}\, 2^{-j(s-\frac{n}{p})}\, h^G_{jm} \tag{2.220}$$

with

$$\mu^G_{jm} = \mu^G_{jm}(f) = 2^{j(s-\frac{n}{p}+n)} \int_{\mathbb{Q}^n} f(x)\, h^G_{jm}(x)\, \mathrm{d}x \tag{2.221}$$

and

$$\|f\,|H^s_p(\mathbb{Q}^n)\| \sim \Big\| \Big(\sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{P}^n_j} |\mu^G_{jm}(f)\, \chi^{(p)}_{jm}(\cdot)|^2 \Big)^{1/2} |L_p(\mathbb{Q}^n)\Big\| \tag{2.222}$$

(*equivalent quasi-norms*).

Proof. This is a special case of Theorem 2.26 and the restriction of Corollary 2.23 to $\mathbb{Q}^n$. □

Remark 2.29. If $s = 0$, then one has again a *Littlewood–Paley characterisation* for

$$L_p(\mathbb{Q}^n) = H_p^0(\mathbb{Q}^n), \quad 1 < p < \infty. \tag{2.223}$$

This is the restriction of (2.194)–(2.197) to $\mathbb{Q}^n$.

2.4 Haar tensor bases on $\mathbb{R}^n$ and on cubes

2.4.1 Introduction

The Haar wavelet system (2.175) is the adequate construction in the context of isotropic spaces $B^s_{pq}(\mathbb{R}^n)$, $F^s_{pq}(\mathbb{R}^n)$ and their restrictions to the unit cube $\mathbb{Q}^n$ in $\mathbb{R}^n$. One of the major topics of this book is the study of spaces with dominating mixed smoothness and their use in connection with numerical integration and related questions. For this purpose one needs *Haar tensor systems* in $\mathbb{R}^n$ and $\mathbb{Q}^n$, $n \geq 2$. Let

$$\big\{h_{jm} : j \in \mathbb{N}_{-1},\, m \in \mathbb{Z}\big\}, \quad \mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}, \tag{2.224}$$

be the (L_∞-normalised) Haar basis in $L_2(\mathbb{R})$ according to (2.93)–(2.96). Let again $\mathbb{N}^n_{-1}$ be the collection of all $k = (k_1, \ldots, k_n)$ with $k_j \in \mathbb{N}_{-1}$ as used in (1.221) in connection with smooth wavelets. Similarly as there we introduce the Haar tensor system

$$\big\{h_{km} : k \in \mathbb{N}^n_{-1},\, m \in \mathbb{Z}^n\big\} \tag{2.225}$$

with

$$h_{km}(x) = \prod_{j=1}^n h_{k_j m_j}(x_j), \quad k \in \mathbb{N}^n_{-1},\, m \in \mathbb{Z}^n, \tag{2.226}$$

preferring now an L_∞-normalisation. We are looking for counterparts of (1.222)–(1.224). Let temporarily $s_{pq} f^-$ be the n-dimensional version of the sequence spaces (1.203), (1.205) (introduced later on in greater detail). Then it follows from Theorem 2.1 in [Kam96] that $\{h_{km}\}$ in (2.225) is an unconditional basis in $L_p(\mathbb{R}^n)$, $1 < p < \infty$,

$$f = \sum_{k \in \mathbb{N}^n_{-1}} \sum_{m \in \mathbb{Z}^n} \lambda_{km}\, 2^{\frac{1}{2}(k_1 + \cdots + k_n)}\, h_{km}, \tag{2.227}$$

$$\lambda_{km} = 2^{\frac{1}{2}(k_1 + \cdots + k_n)} \int_{\mathbb{R}^n} f(x)\, h_{km}(x)\, \mathrm{d}x, \tag{2.228}$$

$$\|f \,|L_p(\mathbb{R}^n)\| \sim \|\lambda \,|s_{p,2} f^-\|. \tag{2.229}$$

This Littlewood–Paley characterisation in terms of Haar tensor systems can be extended to some Sobolev spaces with dominating mixed smoothness $S^r_p H(\mathbb{R}^n) = S^r_{p,2} F(\mathbb{R}^n)$ in

(1.219) and some spaces $S^r_{pp}B(\mathbb{R}^n)$ admitting tensor decompositions. This will be done in Section 2.4.2. Based on other arguments we describe in Section 2.4.3 corresponding assertions for the spaces $S^r_{pq}B(\mathbb{R}^n)$. Section 2.4.4 deals with the restriction of these results to the unit cube $\mathbb{Q}^n$. For sake of simplicity we assume $n = 2$ in Sections 2.4.2-2.4.4, but we indicate in Section 2.4.5 how higher dimensional generalisations look like.

2.4.2 Haar tensor bases on $\mathbb{R}^2$, I

Let $S^r_{pq}B(\mathbb{R}^2)$ and $S^r_{pq}F(\mathbb{R}^2)$ be the spaces with dominating mixed smoothness as introduced in Definition 1.38 where

$$0 < p \le \infty, \quad 0 < q \le \infty, \quad r \in \mathbb{R}, \tag{2.230}$$

(with $p < \infty$ for the F-spaces). Let

$$S^r_p H(\mathbb{R}^2) = S^r_{p,2}F(\mathbb{R}^2), \quad 1 < p < \infty, \ r \in \mathbb{R}, \tag{2.231}$$

according to (1.217)–(1.220) be the corresponding Sobolev spaces with the classical Sobolev spaces

$$S^r_p W(\mathbb{R}^2) = S^r_p H(\mathbb{R}^2), \quad 1 < p < \infty, \ r \in \mathbb{N}_0, \tag{2.232}$$

as a subclass. They can be equivalently normed by

$$\|f \,|S^r_p H(\mathbb{R}^2)\| \sim \big\| \big((1+\xi_1^2)^{r/2}(1+\xi_2^2)^{r/2}\hat{f}\big)^\vee |L_p(\mathbb{R}^2)\big\| \tag{2.233}$$

and

$$\|f \,|S^r_p W(\mathbb{R}^2)\| \sim \sum_{\substack{\alpha\in\mathbb{N}_0^2 \\ 0\le \alpha_1,\alpha_2\le r}} \|D^\alpha f \,|L_p(\mathbb{R}^2)\|. \tag{2.234}$$

We refer in this context to [ST87, Section 2.3.1, pp. 102–104]. We need the sequence spaces $s_{pq}b^-$ and $s_{pq}f^-$ according to (1.203)–(1.205), but for our later purposes it is suitable to change the notation as follows.

Definition 2.30. Let $0 < p, q \le \infty$ and

$$\lambda = \big\{ \lambda_{km} \in \mathbb{C} : k \in \mathbb{N}^2_{-1}, \ m \in \mathbb{Z}^2 \big\}. \tag{2.235}$$

Let χ_{km} be the characteristic function of the rectangle

$$Q_{km} = \big(2^{-k_1}m_1, 2^{-k_1}(m_1+1)\big) \times \big(2^{-k_2}m_2, 2^{-k_2}(m_2+1)\big) \tag{2.236}$$

and

$$\chi^{(p)}_{km}(x) = 2^{\frac{k_1+k_2}{p}} \chi_{km}(x), \quad k \in \mathbb{N}^2_{-1}, \ m \in \mathbb{Z}^2, \ x \in \mathbb{R}^2, \tag{2.237}$$

its p-normalisation. Then $s_{pq}b(\mathbb{R}^2)$ is the collection of all sequences (2.235) with

$$\|\lambda \,|s_{pq}b(\mathbb{R}^2)\| = \Big(\sum_{k\in\mathbb{N}^2_{-1}} \Big(\sum_{m\in\mathbb{Z}^2} |\lambda_{km}|^p \Big)^{q/p} \Big)^{1/q} < \infty \tag{2.238}$$

and $s_{pq}f(\mathbb{R}^2)$ is the collection of all sequences (2.235) with

$$\|\lambda \,|s_{pq}f(\mathbb{R}^2)\| = \Big\| \Big(\sum_{k,m} |\lambda_{km}\chi^{(p)}_{km}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^2) \Big\| < \infty \tag{2.239}$$

with the usual modifications if $p = \infty$ and /or $q = \infty$.

Remark 2.31. This coincides essentially with corresponding definitions in Section 1.2.2 and (1.203)–(1.205). Let

$$\big\{ h_{km} : k \in \mathbb{N}^2_{-1},\ m \in \mathbb{Z}^2 \big\} \tag{2.240}$$

with

$$h_{km}(x) = h_{k_1 m_1}(x_1)\, h_{k_2 m_2}(x_2), \quad k \in \mathbb{N}^2_{-1},\ m \in \mathbb{Z}^2,\ x \in \mathbb{R}^2, \tag{2.241}$$

be as in (2.225), (2.226) with $n = 2$, based on (2.224). We ask for a counterpart of Theorem 1.54 in terms of the Haar tensor basis (2.240) dealing first with the spaces $S^r_{pp}B(\mathbb{R}^2)$ and $S^r_p H(\mathbb{R}^2)$ which are optimally adapted to tensor products. What is meant by unconditional basis, unconditional convergence and local convergence has been explained in Section 1.2.4 with a reference to Section 1.1.4.

Theorem 2.32. (i) *Let*

$$0 < p \le \infty, \quad \frac{1}{p} - 1 < r < \min\Big(\frac{1}{p}, 1\Big), \tag{2.242}$$

Figure 2.3, p. 82 (with $s = r$). Let $f \in S'(\mathbb{R}^2)$. Then $f \in S^r_{pp}B(\mathbb{R}^2)$ if, and only if, it can be represented as

$$f = \sum_{k\in\mathbb{N}^2_{-1}} \sum_{m\in\mathbb{Z}^2} \lambda_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})}\, h_{km}, \quad \lambda \in s_{pp}b(\mathbb{R}^2), \tag{2.243}$$

unconditional convergence being in $S'(\mathbb{R}^2)$ and locally in any space $S^\varrho_{pp}B(\mathbb{R}^2)$ with $\varrho < r$. The representation (2.243) is unique,

$$\lambda_{km} = \lambda_{km}(f) = 2^{(k_1+k_2)(r-\frac{1}{p}+1)} \int_{\mathbb{R}^2} f(x)\, h_{km}(x)\, \mathrm{d}x, \tag{2.244}$$

and

$$J : f \mapsto \{\lambda_{km}(f)\} \tag{2.245}$$

is an isomorphic map of $S^r_{pp}B(\mathbb{R}^2)$ *onto* $s_{pp}b(\mathbb{R}^2)$. *If* $p < \infty$ *then* (2.240) *is an unconditional basis in* $S^r_{pp}B(\mathbb{R}^2)$.

(ii) *Let*

$$\begin{cases} 2 \le p < \infty, & -\frac{1}{2} < r < \frac{1}{p}, \\ 1 < p < 2, & \frac{1}{p} - 1 < r < \frac{1}{2}, \end{cases} \tag{2.246}$$

Figure 2.3, *p.* 82 (*with* $s = r$). *Let* $f \in S'(\mathbb{R}^2)$. *Then* $f \in S^r_p H(\mathbb{R}^2)$ *if, and only if, it can be represented as*

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{Z}^2} \lambda_{km} 2^{-(k_1+k_2)(r-\frac{1}{p})} h_{km}, \quad \lambda \in s_{p,2}f(\mathbb{R}^2). \tag{2.247}$$

This representation is unique with (2.244), J *in* (2.245) *is an isomorphic map of* $S^r_p H(\mathbb{R}^2)$ *onto* $s_{p,2}f(\mathbb{R}^2)$, *and* (2.240) *is an unconditional basis in* $S^r_p H(\mathbb{R}^2)$.

Proof. Step 1. By Theorem 2.9 (i) the Haar system $\{h_{jm}\}$ in (2.224) is a basis in $B^r_{pp}(\mathbb{R})$ (isomorphic map if $p \le \infty$). We apply Theorem 1.58 (i) with $\{h_{jm}\}$ in place of $\{\psi_{jm}\}$ in (1.238). Then one can argue in the same way as in Step 1 of the proof of Theorem 1.66. One obtains the tensor system $\{h_{km}\}$ in (2.240) and the claimed mapping properties.

Step 2. By Theorem 2.9 (ii) and Remark 2.12 the Haar system $\{h_{jm}\}$ in (2.224) is a basis in $H^r_p(\mathbb{R})$ with p, r as in (2.246). Then it follows from [SiU08, Corollary 1] that $\{h_{km}\}$ in (2.240) is a basis in $S^r_p H(\mathbb{R}^2)$ with the indicated properties. □

Remark 2.33. In part (ii) we relied on [SiU08] whereas we reduced part (i) to Theorem 1.58. But one can also use [SiU08, Theorem 2.3] to prove part (i) of the theorem. However all arguments are based on the observation that these spaces are tensor products of corresponding spaces on $\mathbb{R}$. What about the other spaces $S^r_{pq}B(\mathbb{R}^2)$ and $S^r_{pq}F(\mathbb{R}^2)$ with counterparts $B^r_{pq}(\mathbb{R})$ and $F^r_{pq}(\mathbb{R})$ covered by Theorem 2.9 (where $s = r$)? The convenient argument of real interpolation of (isotropic or anisotropic) spaces stepping from B^s_{pp} to B^s_{pq} does not work for spaces with dominating mixed smoothness, [ScS04, Section 4]. But in the next section we describe a direct approach at least for the spaces $S^r_{pq}B(\mathbb{R}^2)$.

2.4.3 Haar tensor bases on $\mathbb{R}^2$, II

We wish to extend part (i) of Theorem 2.32 to all spaces $S^r_{pq}B(\mathbb{R}^2)$ with p, r as in (2.242) and $0 < q \le \infty$. These are the same restrictions for $p, q, s = r$ as in part (i) of Theorem 2.9. For this purpose we extend the crucial Proposition 2.6 (i) from $B^s_{pq}(\mathbb{R})$ to $S^r_{pq}B(\mathbb{R}^2)$. Recall that

$$\chi_{km}(x) = \chi_{k_1m_1}(x_1)\, \chi_{k_2m_2}(x_2), \quad x = (x_1, x_2) \in \mathbb{R}^2, \tag{2.248}$$

are the characteristic functions of the rectangles Q_{km} in (2.236). The sequence spaces $s_{pq}b(\mathbb{R}^2)$ have been introduced in Definition 2.30. As for the convergence of the series below one may consult the comments and references in front of Proposition 2.6.

Proposition 2.34. *Let* $0 < p, q \le \infty$ *and*

$$\max\left(\frac{1}{p}, 1\right) - 1 < r < \frac{1}{p}. \tag{2.249}$$

Then

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{Z}^2} \mu_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})}\, \chi_{km}, \quad \mu \in s_{pq}b(\mathbb{R}^2), \tag{2.250}$$

belongs to $S^r_{pq}B(\mathbb{R}^2)$ *and*

$$\| f \,|S^r_{pq}B(\mathbb{R}^2)\| \le c\, \|\mu\, |s_{pq}b(\mathbb{R}^2)\| \tag{2.251}$$

for some $c > 0$ *and all* $\mu \in s_{pq}b(\mathbb{R}^2)$.

Proof. We follow the relevant parts of the proof of Proposition 2.6. First one expands $\chi_{k_1m_1}(x_1)$ and $\chi_{k_2m_2}(x_2)$ in (2.248) as in (2.51)–(2.55) and inserts the outcome in (2.250). We split the resulting expansions as in (2.56)–(2.60), where one has now four terms related to

$$(j_1 \ge k_1, j_2 \ge k_2), \quad (j_1 \ge k_1, j_2 < k_2), \quad (j_1 < k_1, j_2 \ge k_2), \quad (j_1 < k_1, j_2 < k_2).$$

As far as the coefficients in front of μ_{km} are concerned one has in all four cases an exponential decay in both directions. This applies to the counterparts of (2.62)–(2.66) and of (2.73), (2.74). In other words, the exponential decay in both directions of the factors of μ_{km} ensures that the B-part of the proof of Proposition 2.6 (i) can be carried over resulting in (2.251). □

Remark 2.35. Let $s_{pq}f(\mathbb{R}^2)$ be the sequence spaces as introduced in Definition 2.30. There is little doubt that also part (ii) of Proposition 2.6 can be extended to the above situation. This means that

for $0 < p, q < \infty$ *and*

$$\max\left(\frac{1}{p}, \frac{1}{q}, 1\right) - 1 < r < \min\left(\frac{1}{p}, \frac{1}{q}\right) \tag{2.252}$$

any f *which can be represented as*

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{Z}^2} \mu_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})}\, \chi_{km}, \quad \mu \in s_{pq}f(\mathbb{R}^2), \tag{2.253}$$

belongs to $S^r_{pq}F(\mathbb{R}^2)$ *with*

$$\| f \,|S^r_{pq}F(\mathbb{R}^2)\| \le c\, \|\mu\, |s_{pq}f(\mathbb{R}^2)\| \tag{2.254}$$

for some $c > 0$ *and all* $\mu \in s_{pq}f(\mathbb{R}^2)$.

One must find suitable substitutes of the relevant parts of the proof of Proposition 2.6. First we remark that there is a coordinate-wise generalisation of the vector-valued maximal inequality (2.67) which can be found in [ST87, p. 23] with a reference to [Bag75]. This has been used in connection with spaces with dominating mixed smoothness in [Schm07], [Vyb06] and also in the proof of Theorem 1.52. This means that one can adapt (2.67)–(2.70) to the above situation. The complex interpolation (2.83), (2.84) according to [MeM00], [KMM07] has a counterpart for the spaces $s_{pq}f(\mathbb{R}^2)$ and $s_{pq}b(\mathbb{R}^2)$. We refer to [Vyb06, Section 4]. In any case there is a good chance to prove (2.252)–(2.254) which might be considered as a well-supported ***conjecture***. Of special interest would be again the (Hardy–)Sobolev spaces with dominating mixed smoothness,

$$S^r_p H(\mathbb{R}^2) = S^r_{p,2} F(\mathbb{R}^2), \quad 0 < p < \infty, \ r \in \mathbb{R}, \tag{2.255}$$

in generalisation of (2.231) and (1.217)–(1.220).

Remark 2.36. There is a counterpart of Remark 2.7. Recall that χ_{km} is the characteristic function of the rectangle

$$Q_{km} = \big(2^{-k_1}m_1, 2^{-k_1}(m_1+1)\big) \times \big(2^{-k_2}m_2, 2^{-k_2}(m_2+1)\big) = I_{k_1m_1} \times I_{k_2m_2}. \tag{2.256}$$

Then one can replace χ_{km} in (2.250) (and (2.253)) by polynomials

$$P_{km}(x) = P_{k_1m_1}(x_1)\, P_{k_2m_2}(x_2), \quad x \in Q_{km}, \tag{2.257}$$

with

$$\text{degree } P_{k_jm_j}(x_j) \le K, \quad \sup_{x_j \in I_{k_jm_j}} |P_{k_jm_j}(x_j)| \le 1, \tag{2.258}$$

where $j = 1, 2$. This follows from the arguments in Remark 2.7 and the above proof.

Next we need the counterpart of Proposition 2.8. We rely on Theorem 1.52 and the notation introduced there. Let $\{h_{km}\}$ be the Haar system in (2.240), (2.241) based on the rectangles $Q_{km} = I_{k_1m_1} \times I_{k_2m_2}$ in (2.256). Let

$$W_{km}(f) = \int_{\mathbb{R}^2} 2^{k_1+k_2}\, h_{km}(x)\, f(x)\, \mathrm{d}x, \quad k \in \mathbb{N}^2_{-1},\ m \in \mathbb{Z}^2, \tag{2.259}$$

be the local means according to Definitions 1.46, 1.50 with $A = 0$ and $B = 1$. Let

$$\mu(f) = \{\mu_{km}(f)\} \quad \text{with } \mu_{km}(f) = 2^{(k_1+k_2)(r-\frac{1}{p})}\, W_{km}(f) \tag{2.260}$$

where $0 < p \le \infty$, $r \in \mathbb{R}$, $k \in \mathbb{N}^2_{-1}$, $m \in \mathbb{Z}^2$. The sequence spaces $s_{pq}b(\mathbb{R}^2)$ and $s_{pq}f(\mathbb{R}^2)$ have been introduced in Definition 2.30.

Proposition 2.37. (i) *Let* $0 < p, q \le \infty$ *and*

$$\max\Big(\frac{1}{p}, 1\Big) - 1 < r < 1. \tag{2.261}$$

Then there is a $c > 0$ *such that*

$$\|\mu(f)\,|s_{pq}b(\mathbb{R}^2)\| \le c\,\|f\,|S^r_{pq}B(\mathbb{R}^2)\| \quad \textit{for all } f \in S^r_{pq}B(\mathbb{R}^2). \tag{2.262}$$

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$ *and*

$$\max\Big(\frac{1}{p},\frac{1}{q},1\Big) - 1 < r < 1. \tag{2.263}$$

Then there is a $c > 0$ *such that*

$$\|\mu(f)\,|s_{pq}f(\mathbb{R}^2)\| \le c\,\|f\,|S^r_{pq}F(\mathbb{R}^2)\| \quad \textit{for all } f \in S^r_{pq}F(\mathbb{R}^2). \tag{2.264}$$

Proof. By the above considerations one can apply Theorem 1.52 with $A = 0$ and $B = 1$. Then one has (1.192), (1.194) with respect to the sequence spaces $s^r_{pq}b$ and $s^r_{pq}f$ where $k \in \mathbb{N}^2_0$ is replaced now by $k \in \mathbb{N}^2_{-1}$. But this can be rewritten as (2.262), (2.264). □

Now we are in the position to extend Theorem 2.32 (i) from $S^r_{pp}B(\mathbb{R}^2)$ to $S^r_{pq}B(\mathbb{R}^2)$ adapting the arguments in the proof of Theorem 2.9 (i) to the above situation. As for technicalities one may consult the references in front of Theorem 2.32 and in Step 1 of the proof of Theorem 2.9. Let again

$$\big\{h_{km} : k \in \mathbb{N}^2_{-1},\ m \in \mathbb{Z}^2\big\} \tag{2.265}$$

be the (L_∞-normalised) Haar tensor system (2.225), (2.226) with $n = 2$, based on (2.224). Let $s_{pq}b(\mathbb{R}^2)$ be the sequence spaces introduced in Definition 2.30.

Theorem 2.38. *Let* $0 < p \le \infty$, $0 < q \le \infty$ ($1 < q \le \infty$ *if* $p = \infty$), *and*

$$\frac{1}{p} - 1 < r < \min\Big(\frac{1}{p},1\Big), \tag{2.266}$$

Figure 2.3, *p.* 82 (*with* $s = r$). *Let* $f \in S'(\mathbb{R}^2)$. *Then* $f \in S^r_{pq}B(\mathbb{R}^2)$ *if, and only if, it can be represented as*

$$f = \sum_{k\in\mathbb{N}^2_{-1}} \sum_{m\in\mathbb{Z}^2} \mu_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})}\, h_{km}, \quad \mu \in s_{pq}b(\mathbb{R}^2), \tag{2.267}$$

unconditional convergence being in $S'(\mathbb{R}^2)$ *and locally in any space* $S^\varrho_{pq}B(\mathbb{R}^2)$ *with* $\varrho < r$. *The representation* (2.267) *is unique,*

$$\mu_{km} = \mu_{km}(f) = 2^{(k_1+k_2)(r-\frac{1}{p}+1)} \int_{\mathbb{R}^2} f(x)\, h_{km}(x)\, \mathrm{d}x \tag{2.268}$$

and

$$J : f \mapsto \{\mu_{km}(f)\} \tag{2.269}$$

is an isomorphic map of $S^r_{pq}B(\mathbb{R}^2)$ *onto* $s_{pq}b(\mathbb{R}^2)$. *If* $p < \infty$, $q < \infty$, *then* (2.265) *is an unconditional basis in* $S^r_{pq}B(\mathbb{R}^2)$.

Proof. Step 1. Let $0 < p, q \le \infty$ and

$$\max\left(\frac{1}{p}, 1\right) - 1 < r < \min\left(\frac{1}{p}, 1\right). \tag{2.270}$$

Let f be given by (2.267). Then it follows from Proposition 2.34 that $f \in S^r_{pq}B(\mathbb{R}^2)$ with (2.251). Conversely if $f \in S^r_{pq}B(\mathbb{R}^2)$ then one has (2.262). If, in addition, f can be represented by (2.267), (2.268) then one obtains

$$\| f \,|S^r_{pq}B(\mathbb{R}^2)\| \sim \|\mu(f)\,|s_{pq}b(\mathbb{R}^2)\|. \tag{2.271}$$

But this representability follows from the corresponding assertion in Theorem 2.32. (It is essentially a consequence of the observation that $\{h_{km}\}$ is an orthogonal basis in $L_2(\mathbb{R}^2)$ combined with the technicalities mentioned in Step 1 of the proof of Theorem 2.9).

Step 2. The counterpart of the duality (1.76) is given by

$$S^r_{pq}B(\mathbb{R}^2)' = S^{-r}_{p'q'}B(\mathbb{R}^2), \tag{2.272}$$

$$r \in \mathbb{R}, \quad 1 \le p < \infty, \quad 1 \le q < \infty, \quad \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1. \tag{2.273}$$

One can prove this assertion by reduction to sequence spaces according to the wavelet isomorphisms in Theorem 1.54 (where one has to choose $u \in \mathbb{N}$ sufficiently large). The corresponding duality for sequence spaces can be obtained in the standard way. Then one is in the same situation as in Step 3 of the proof of Theorem 2.9. We apply (2.272) to the spaces $S^r_{pq}B(\mathbb{R}^2)$ with $1 \le p, q < \infty$, $0 < r < 1/p$, which are covered by Step 1. This gives the desired assertion for the spaces $S^r_{pq}B(\mathbb{R}^2)$ with $1 < p, q \le \infty$ and $\frac{1}{p} - 1 < r < 0$.

Step 3. The remaining cases with $q < \infty$ can be obtained by real interpolation

$$\big(S^{r_0}_{pq_0}B(\mathbb{R}^2), S^{r_1}_{pq_1}B(\mathbb{R}^2)\big)_{\theta,q} = S^r_{pq}B(\mathbb{R}^2) \tag{2.274}$$

with $1 < p < \infty$, $0 < \theta < 1$,

$$0 < r_0 < \frac{1}{p}, \quad \frac{1}{p} - 1 < r_1 < 0, \quad 0 < q_0 < \infty, \quad 1 < q_1 < \infty, \tag{2.275}$$

and

$$r = (1-\theta)r_0 + \theta r_1, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{2.276}$$

The spaces on the left-hand side of (2.274) are covered by the Steps 1 and 2. Any other space of the theorem with $p < \infty$, $q < \infty$ can be reached in this way. The interpolation (2.274) restricted to Banach spaces, hence $q_0 \ge 1$, may be found in [ScS04, Proposition 7, p. 127]. Using the isomorphism according to Theorem 1.54 one can reduce the proof of (2.274) to an interpolation formula of type

$$\big(\ell_{q_0}(2^{jr_0}\ell_p), \ell_{q_1}(2^{jr_1}\ell_p)\big)_{\theta,q} = \ell_q(2^{jr}\ell_p) \tag{2.277}$$

which is covered by [T78], Theorem on p. 121 (Banach spaces) and Remark 4, p. 123 (quasi-Banach spaces). Finally the cases

$$1 < p < \infty, \quad \frac{1}{p} - 1 < r \leq 0, \quad q = \infty,$$

can be incorporated by a duality argument as in Step 2. □

Remark 2.39. The above theorem applies to the same parameters p, q, r for the spaces $S^r_{pq}B(\mathbb{R}^2)$ as Theorem 2.9 (i) to $p, q, s = r$ for the spaces $B^s_{pq}(\mathbb{R})$ (with exception of $p = \infty$, $0 < q \leq 1$). One can expected (and ***conjecture***) that one has the same situation for the spaces $S^r_{pq}F(\mathbb{R}^2)$ compared with the spaces $F^s_{pq}(\mathbb{R})$ covered by part (ii) of Theorem 2.9. This is supported by our comments in Remark 2.35 and by Proposition 2.37 which applies both to B-spaces and F-spaces. These two observations are the crucial ingredients for the proof of Theorem 2.38. Of special interest are the (Hardy–)Sobolev spaces with dominating mixed smoothness

$$S^r_p H(\mathbb{R}^2) = S^r_{p,2}F(\mathbb{R}^2), \quad 0 < p < \infty, \ r \in \mathbb{R}. \tag{2.278}$$

In other words, one can expect that Theorem 2.32 (ii) can be extended to all spaces $S^r_p H(\mathbb{R}^2)$ with

$$\begin{cases} 2 \leq p < \infty, & -\frac{1}{2} < r < \frac{1}{p}, \\ 0 < p < 2, & \frac{1}{p} - 1 < r < \frac{1}{2}, \end{cases} \tag{2.279}$$

in agreement with Remark 2.12, (2.127), and Figure 2.3, p. 82 (with $s = r$).

2.4.4 Haar tensor bases on cubes

Let $\mathbb{Q}^n$ be the unit cube (2.202) on $\mathbb{R}^n$, $2 \leq n \in \mathbb{N}$. Let $S^r_{pq}B(\mathbb{Q}^n)$ and $S^r_{pq}F(\mathbb{Q}^n)$ be the corresponding spaces with dominating mixed smoothness as introduced in Definition 1.56. Let

$$S^r_p H(\mathbb{Q}^n) = S^r_{p,2}F(\mathbb{Q}^n), \quad 0 < p < \infty, \ r \in \mathbb{R}, \tag{2.280}$$

be the (Hardy–)Sobolev spaces on $\mathbb{Q}^n$. Similarly as in Section 2.3.3 with respect to the *Haar wavelet system* on $\mathbb{Q}^n$ we ask now in which of the above spaces with dominating mixed smoothness the restriction of the *Haar tensor system* (2.225), (2.226) generates a basis. As before we assume $n = 2$, adding later on a few comments about higher dimensions. Let

$$\mathbb{Q}^2 = \big\{ x = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, \ 0 < x_2 < 1 \big\} \tag{2.281}$$

be the unit square in $\mathbb{R}^2$. We adapt the restriction of (2.225), (2.226) with $n = 2$ to the Haar bases on the interval $I = (0, 1)$ according to Section 2.2.4 and to our later notation in connection with Faber bases. Let as there

$$\big\{ h_0, \ h_{jm} : j \in \mathbb{N}_0; \ m = 0, \ldots, 2^j - 1 \big\} \tag{2.282}$$

be the L_∞-normalised orthogonal Haar system in $L_2(I)$ where h_0 is the characteristic function of I and

$$h_{jm}(t) = \begin{cases} 1 & \text{if } 2^{-j}m \le t < 2^{-j}m + 2^{-j-1}, \\ -1 & \text{if } 2^{-j}m + 2^{-j-1} \le t < 2^{-j}(m+1), \\ 0 & \text{otherwise.} \end{cases} \tag{2.283}$$

Then the Haar tensor system on $\mathbb{Q}^2$ obtained by restriction of (2.225), (2.226) to $\mathbb{Q}^2$ is given by

$$h_{km}(x) = \begin{cases} 2h_0(x_1)h_0(x_2) \\ \qquad \text{if } k = (-1,-1), m = (0,0), \\ \sqrt{2}\, h_0(x_1)h_{k_2m_2}(x_2) \\ \qquad \text{if } k = (-1,k_2), k_2 \in \mathbb{N}_0, m = (0,m_2), m_2 = 0,\ldots,2^{k_2}-1, \\ \sqrt{2}\, h_{k_1m_1}(x_1)h_0(x_2) \\ \qquad \text{if } k = (k_1,-1), k_1 \in \mathbb{N}_0, m = (m_1,0), m_1 = 0,\ldots,2^{k_1}-1, \\ h_{k_1m_1}(x_1)h_{k_2m_2}(x_2) \\ \qquad \text{if } k \in \mathbb{N}_0^2, m_l = 0,\ldots,2^{k_l}-1; l = 1,2. \end{cases} \tag{2.284}$$

Recall $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\} = \{-1,0,1,\ldots\}$ and $\mathbb{N}_{-1}^2 = \mathbb{N}_{-1} \times \mathbb{N}_{-1}$. Let

$$\mathbb{P}_k^H = \{m \in \mathbb{Z}^2 \text{ with } m \text{ as in (2.284)}\}, \quad k \in \mathbb{N}_{-1}^2. \tag{2.285}$$

Then

$$\{h_{km} : k \in \mathbb{N}_{-1}^2,\ m \in \mathbb{P}_k^H\} \tag{2.286}$$

is the *Haar tensor system* in $\mathbb{Q}^2$ we are looking for and

$$\left\{2^{\frac{k_1+k_2}{2}} h_{km} : k \in \mathbb{N}_{-1}^2,\ m \in \mathbb{P}_k^H\right\} \tag{2.287}$$

is an orthonormal basis in $L_2(\mathbb{Q}^2)$. We need the counterparts of the sequence spaces $b_{pq}(I)$, $f_{pq}(I)$ in (2.131), (2.132) which are the restrictions of the sequence spaces in Definition 2.30 to $\mathbb{Q}^2$. One can replace $\chi_{km}^{(p)}(x)$ in (2.239) and Theorem 2.32 by

$$|h_{km}(x)|^{(p)} = 2^{\frac{k_1+k_2}{p}} |h_{km}(x)|, \quad x \in \mathbb{R}^2. \tag{2.288}$$

This is an immaterial modification of the same type as discussed after (2.207).

Definition 2.40. Let $0 < p, q \le \infty$ and

$$\lambda = \{\lambda_{km} \in \mathbb{C} : k \in \mathbb{N}_{-1}^2,\ m \in \mathbb{P}_k^H\}. \tag{2.289}$$

Then $s_{pq}^H b(\mathbb{Q}^2)$ is the collection of all sequences (2.289) with

$$\|\lambda \,|s_{pq}^H b(\mathbb{Q}^2)\| = \Big(\sum_{k \in \mathbb{N}_{-1}^2} \Big(\sum_{m \in \mathbb{P}_k^H} |\lambda_{km}|^p \Big)^{q/p} \Big)^{1/q} < \infty \tag{2.290}$$

and $s^H_{pq} f(\mathbb{Q}^2)$ is the collection of all sequences (2.289) with

$$\|\lambda \,|s^H_{pq} f(\mathbb{Q}^2)\| = \Big\| \Big(\sum_{k,m} |\lambda_{km}\, |h_{km}(\cdot)|^{(p)}|^q \Big)^{1/q} |L_p(\mathbb{Q}^2) \Big\| < \infty \tag{2.291}$$

with the usual modifications if $p = \infty$ and/or $q = \infty$.

After these preparations one can now restrict Theorems 2.32, 2.38 from $\mathbb{R}^2$ to $\mathbb{Q}^2$. As far as technicalities are concerned we refer to the comments in front of these theorems.

Theorem 2.41. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$ $(1 < q \le \infty$ *if* $p = \infty)$ *and*

$$\frac{1}{p} - 1 < r < \min\Big(\frac{1}{p}, 1\Big), \tag{2.292}$$

Figure 2.3, p. 82 (with $r = s$*). Let* $f \in D'(\mathbb{Q}^2)$*. Then* $f \in S^r_{pq}B(\mathbb{Q}^2)$ *if, and only if, it can be represented as*

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^H_k} \lambda_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})} h_{km}, \quad \lambda \in s^H_{pq} b(\mathbb{Q}^2), \tag{2.293}$$

unconditional convergence being in $D'(\mathbb{Q}^2)$ *and in any spaces* $S^{\varrho}_{pq}B(\mathbb{Q}^2)$ *with* $\varrho < r$*. The representation (2.293) is unique,* $\lambda = \lambda(f)$ *with*

$$\lambda_{km} = \lambda_{km}(f) = 2^{(k_1+k_2)(r-\frac{1}{p}+1)} \int_{\mathbb{Q}^2} f(x)\, h_{km}(x)\, \mathrm{d}x \tag{2.294}$$

and

$$J: \ f \mapsto \lambda(f) \tag{2.295}$$

is an isomorphic map of $S^r_{pq}B(\mathbb{Q}^2)$ *onto* $s^H_{pq}b(\mathbb{Q}^2)$*. If, in addition,* $p < \infty$, $q < \infty$*, then* $\{h_{km}\}$ *is an unconditional basis in* $S^r_{pq}B(\mathbb{Q}^2)$*.*

(ii) *Let*

$$\begin{cases} 2 \le p < \infty, & -\frac{1}{2} < r < \frac{1}{p}, \\ 1 < p < 2, & \frac{1}{p} - 1 < r < \frac{1}{2}, \end{cases} \tag{2.296}$$

Figure 2.3, p. 82 (with $r = s$*). Let* $f \in D'(\mathbb{Q}^2)$*. Then* $f \in S^r_p H(\mathbb{Q}^2)$ *if, and only if, it can be represented as*

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^H_k} \lambda_{km} 2^{-(k_1+k_2)(r-\frac{1}{p})} h_{km}, \quad \lambda \in s^H_{p,2} f(\mathbb{Q}^2), \tag{2.297}$$

unconditional convergence being in $S^r_p H(\mathbb{Q}^2)$*. The representation (2.297) is unique with (2.294). Furthermore,* J *in (2.295) is an isomorphic map of* $S^r_p H(\mathbb{Q}^2)$ *onto* $s^H_{p,2} f(\mathbb{Q}^2)$ *and* $\{h_{km}\}$ *is an unconditional basis in* $S^r_p H(\mathbb{Q}^2)$*.*

Proof. We prove part (i). Let $0 < p, q \le \infty$ and

$$\max\left(\frac{1}{p}, 1\right) - 1 < r < \min\left(\frac{1}{p}, 1\right). \tag{2.298}$$

Let

$$\widetilde{S}^r_{pq}B(\mathbb{Q}^2) = \left\{f \in S^r_{pq}B(\mathbb{R}^2) : \operatorname{supp} f \subset \overline{\mathbb{Q}^2}\right\}. \tag{2.299}$$

Then any $f \in \widetilde{S}^r_{pq}B(\mathbb{Q}^2)$ can be represented by (2.267) with $m \in \mathbb{P}^H_k$ in place of $m \in \mathbb{Z}^2$. Furthermore one has

$$\widetilde{S}^r_{pq}B(\mathbb{Q}^2) \hookrightarrow L_1(\mathbb{Q}^2)$$

by elementary embedding, [ST87, Section 2.2.3]. Both together show that the characteristic function of $\mathbb{Q}^2$ is a pointwise multiplier in $S^r_{pq}B(\mathbb{Q}^2)$. This can be extended by duality and interpolation to all spaces $S^r_{pq}B(\mathbb{Q}^2)$ in part (i) in the same way as in the proof of Theorem 2.38. Then one can identify $S^r_{pq}B(\mathbb{Q}^2)$ with $\widetilde{S}^r_{pq}B(\mathbb{Q}^2)$ and part (i) follows from Theorem 2.38. By the same arguments one obtains part (ii) from Theorem 2.32 (ii). □

Remark 2.42. We restricted the above theorem to spaces covered by Theorems 2.32, 2.38. Otherwise the conditions for the parameters $p, q, r = s$ are the same as for Haar bases on intervals and on $\mathbb{R}$ according to Theorem 2.13 and Remark 2.12, Figure 2.3, p. 82 (in contrast to Theorem 2.26 and Corollary 2.28, Figure 2.4, p. 94). But we conjectured in Remark 2.39 that Theorem 2.38 can be extended to all spaces $S^r_{pq}F(\mathbb{R}^2)$ with $p, q, s = r$ as in Theorem 2.9 (ii). Then Theorem 2.41 (ii) can also be extended to $S^r_{pq}F(\mathbb{Q}^2)$ with $p, q, r = s$ as in (2.112). Haar tensor systems in squares in $\mathbb{R}^2$ and in cubes in $\mathbb{R}^n$ have been used in [Osw99] to study N-term approximations in spaces of type $S^r_{2,2}B(Q)$ denoted there as $H^r_{\mathrm{mix}}(Q)$.

2.4.5 Higher dimensions

We introduced in Section 1.2.5 the spaces $S^r_{pq}B(\mathbb{R}^n)$, $S^r_{pq}F(\mathbb{R}^n)$ and their special cases $S^r_pH(\mathbb{R}^2)$, $S^r_pW(\mathbb{R}^n)$ in higher dimensions and described in (1.221)–(1.224) related representations in terms of compactly supported smooth wavelets. There are more or less obvious counterparts for Haar bases extending the above assertions from two to $2 \le n \in \mathbb{N}$ dimensions. We give a brief description. Let again

$$\left\{h_{jm} : j \in \mathbb{N}_{-1},\ m \in \mathbb{Z}\right\}, \quad \mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}, \tag{2.300}$$

be the (L_∞-normalised) Haar basis in $L_2(\mathbb{R})$ according to (2.93)–(2.96) and let

$$\left\{h_{km} : k \in \mathbb{N}^n_{-1},\ m \in \mathbb{Z}^n\right\} \tag{2.301}$$

with

$$h_{km}(x) = \prod_{j=1}^{n} h_{k_j m_j}(x_j), \quad k \in \mathbb{N}_{-1}^n,\ m \in \mathbb{Z}^n, \tag{2.302}$$

be the same (L_∞-normalised) orthogonal Haar tensor basis in $L_2(\mathbb{R}^n)$ as in (2.225), (2.226). Let χ_{km} be the characteristic function of the rectangle

$$Q_{km} = \big(2^{-k_1} m_1, 2^{-k_1}(m_1+1)\big) \times \cdots \times \big(2^{-k_n} m_n, 2^{-k_n}(m_n+1)\big), \quad k \in \mathbb{N}_{-1}^n,\ m \in \mathbb{Z}^n, \tag{2.303}$$

p-normalised by

$$\chi_{km}^{(p)}(x) = 2^{\frac{1}{p}(k_1+\cdots+k_n)} \chi_{km}(x), \quad x \in \mathbb{R}^n. \tag{2.304}$$

Let $0 < p, q \le \infty$. Then $s_{pq}b(\mathbb{R}^n)$ is the collection of all sequences

$$\lambda = \big\{ \lambda_{km} \in \mathbb{C} : k \in \mathbb{N}_{-1}^n,\ m \in \mathbb{Z}^n \big\} \tag{2.305}$$

with

$$\|\lambda \,|s_{pq}b(\mathbb{R}^n)\| = \Big(\sum_{k \in \mathbb{N}_{-1}^n} \Big(\sum_{m \in \mathbb{Z}^n} |\lambda_{km}|^p \Big)^{q/p} \Big)^{1/q} < \infty \tag{2.306}$$

and $s_{pq}f(\mathbb{R}^n)$ is the collection of all sequences (2.305) with

$$\|\lambda \,|s_{pq}f(\mathbb{R}^n)\| = \Big\| \Big(\sum_{k,m} |\lambda_{km}\, \chi_{km}^{(p)}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{R}^n) \Big\| < \infty. \tag{2.307}$$

This is the obvious generalisation of Definition 2.30.

For later purposes we outline the n-dimensional modifications of Theorems 2.32, 2.38. Let $0 < p, q \le \infty$ ($1 < q \le \infty$ if $p = \infty$) and

$$\frac{1}{p} - 1 < r < \min\Big(\frac{1}{p}, 1\Big). \tag{2.308}$$

Then Theorem 2.38 can be extended to $\mathbb{R}^n$ with the substitutes

$$f = \sum_{k \in \mathbb{N}_{-1}^n} \sum_{m \in \mathbb{Z}^n} \mu_{km}\, 2^{-(k_1+\cdots+k_n)(r-\frac{1}{p})} h_{km}, \quad \mu \in s_{pq}b(\mathbb{R}^n), \tag{2.309}$$

$$\mu_{km} = \mu_{km}(f) = 2^{(k_1+\cdots+k_n)(r-\frac{1}{p}+1)} \int_{\mathbb{R}^n} f(x)\, h_{km}(x)\, \mathrm{d}x \tag{2.310}$$

in place of (2.267), (2.268). Let

$$\begin{cases} 2 \le p < \infty, & -\frac{1}{2} < r < \frac{1}{p}, \\ 1 < p < 2, & \frac{1}{p} - 1 < r < \frac{1}{2}. \end{cases} \tag{2.311}$$

Then Theorem 2.32 (ii) can be extended to $\mathbb{R}^n$ with the substitutes

$$f = \sum_{k \in \mathbb{N}^n_{-1}} \sum_{m \in \mathbb{Z}^n} \mu_{km} \, 2^{-(k_1+\cdots+k_n)(r-\frac{1}{p})} \, h_{km}, \quad \mu \in s_{p,2} f(\mathbb{R}^n), \tag{2.312}$$

and (2.310) in place of (2.247), (2.244).

Finally we add a comment about the n-dimensional version of Theorem 2.41. Let again

$$\mathbb{Q}^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_l < 1\} \tag{2.313}$$

be the unit cube in $\mathbb{R}^n$. Let $S^r_{pq} B(\mathbb{Q}^n)$ and $S^r_{pq} F(\mathbb{Q}^n)$ be the spaces according to Definition 1.56 with $\Omega = \mathbb{Q}^n$ and let $S^r_p H(\mathbb{Q}^n)$ be as in (2.280). Let

$$\big\{ h_0, \, h_{jm} : j \in \mathbb{N}_0; \; m = 0, \ldots, 2^j - 1 \big\} \tag{2.314}$$

with h_{jm} as in (2.283) and the characteristic function h_0 of the interval $I = (0,1)$ be the same L_∞-normalised orthogonal Haar system in $L_2(I)$ as in (2.282). Let h_{km} be the n-dimensional generalisation of (2.284) with the counterpart $\mathbb{P}^{H,n}_k$ of $\mathbb{P}^{H,2}_k = \mathbb{P}^H_k$ in (2.285) for $k \in \mathbb{N}^n_{-1}$, where we indicated temporarily n. Then

$$\Big\{ 2^{\frac{1}{2}(k_1+\cdots+k_n)} \, h_{km} : k \in \mathbb{N}^n_{-1}, \; m \in \mathbb{P}^{H,n}_k \Big\} \tag{2.315}$$

is an orthonormal basis in $L_2(\mathbb{Q}^n)$. Similarly as in Definition 2.40 the space $s^H_{pq} b(\mathbb{Q}^n)$ is the collection of all sequences

$$\lambda = \big\{ \lambda_{km} \in \mathbb{C} : k \in \mathbb{N}^n_{-1}, \; m \in \mathbb{P}^{H,n}_k \big\} \tag{2.316}$$

with

$$\| \lambda \, | s^H_{pq} b(\mathbb{Q}^n) \| = \bigg(\sum_{k \in \mathbb{N}^n_{-1}} \Big(\sum_{m \in \mathbb{P}^{H,n}_k} |\lambda_{km}|^p \Big)^{q/p} \bigg)^{1/q} < \infty. \tag{2.317}$$

Similarly for the spaces $s^H_{pq} f(\mathbb{Q}^n)$ with an obvious counterpart of (2.291). Let $0 < p, q \le \infty$ ($1 < q \le \infty$ if $p = \infty$) and let r be as in (2.292) = (2.308). Then Theorem 2.41 (i) can be extended to $\mathbb{Q}^n$ with the substitutes

$$f = \sum_{k \in \mathbb{N}^n_{-1}} \sum_{m \in \mathbb{P}^{H,n}_k} \lambda_{km} 2^{-(k_1+\cdots+k_n)(r-\frac{1}{p})} \, h_{km}, \quad \lambda \in s^H_{pq} b(\mathbb{Q}^n), \tag{2.318}$$

$$\lambda_{km} = \lambda_{km}(f) = 2^{(k_1+\cdots+k_n)(r-\frac{1}{p}+1)} \int_{\mathbb{Q}^n} f(x) \, h_{km}(x) \, \mathrm{d}x \tag{2.319}$$

in place of (2.293), (2.294). Let p, r be as in (2.296) = (2.311). Then Theorem 2.41 (ii) can be extended to $\mathbb{Q}^n$ with the substitutes

$$f = \sum_{k \in \mathbb{N}^n_{-1}} \sum_{m \in \mathbb{P}^{H,n}_k} \lambda_{km} \, 2^{-(k_1+\cdots+k_n)(r-\frac{1}{p})} h_{km}, \quad \lambda \in s^H_{p,2} f(\mathbb{Q}^n), \tag{2.320}$$

and (2.319) in place of (2.297) and (2.294).

2.5 Spline bases

2.5.1 Preliminaries and basic assertions

The techniques developed in connection with Haar bases work also for some orthogonal spline systems applied to the spaces $B^s_{pq}(\mathbb{R})$, $F^s_{pq}(\mathbb{R})$ on the real line $\mathbb{R}$, their isotropic generalisations $B^s_{pq}(\mathbb{R}^n)$, $F^s_{pq}(\mathbb{R}^n)$ and related spaces $S^r_{pq}B(\mathbb{R}^n)$, $S^r_{pq}F(\mathbb{R}^n)$ with dominating mixed smoothness. This might be of some interest for its own sake. However it does not play a central role later on in this book. We insert some related assertions, but we will be brief. First we describe the background.

The Haar system (2.96), based on (2.93)–(2.95) is now considered as a spline system of order zero. Otherwise we stick at this notation combined with a suitable modification of the standard set-up (1.55), (1.56) for wavelets. Let $l \in \mathbb{N}_0$. Of interest is a *real scaling function* $h^l_F(x)$ and an associated *real wavelet* $h^l_M(x)$ on $\mathbb{R}$ in the context of a related multiresolution analysis with the following properties:

(i) The functions h^l_F, h^l_M have classical continuous derivatives up to order $l-1$ inclusively on $\mathbb{R}$ (no condition if $l = 0$).

(ii) The restriction of h^l_F, h^l_M to each interval $(m, m+\frac{1}{2})$ with $2m \in \mathbb{Z}$ is a polynomial of degree at most l.

(iii) There are constants $c > 0$, $\alpha > 0$, such that

$$\left|\frac{d^k}{dx^k} h^l_F(x)\right| + \left|\frac{d^k}{dx^k} h^l_M(x)\right| \le c\, e^{-\alpha|x|}, \quad 2x \in \mathbb{R}\setminus\mathbb{Z}, \tag{2.321}$$

and $k = 0, \dots, l$.

(iv) For $k = 0, \dots, l$,

$$\int_{\mathbb{R}} x^k\, h^l_M(x)\,\mathrm{d}x = 0. \tag{2.322}$$

The Haar functions $h^0_F = h_F$ and $h^0_M = h_M$ according to (2.93), (2.94) are an example of the above conditions with $l = 0$. In analogy to (2.94), (2.95) we put

$$h^l_{-1,m}(x) = \sqrt{2}\, h^l_F(x-m) \quad \text{and} \quad h^l_{jm}(x) = h^l_M\big(2^j x - m\big), \tag{2.323}$$

where $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}$. Recall that $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$.

Proposition 2.43. *For each* $l \in \mathbb{N}_0$ *there are functions* h^l_F *and* h^l_M *with the above properties* (i)–(iv) *such that*

$$\big\{2^{j/2} h^l_{jm} : j \in \mathbb{N}_{-1},\ m \in \mathbb{Z}\big\} \tag{2.324}$$

is an orthonormal basis in $L_2(\mathbb{R})$.

Remark 2.44. If $l = 0$ then one can take the classical Haar basis (2.96). A detailed streamlined proof of this fundamental assertion for $l \in \mathbb{N}$ may be found in [Woj97, Sections 3.3.1–3.3.3, pp. 52–61]. But we add two comments. The exponential decay (2.321) is stated in [Woj97, Proposition 3.17, Theorem 3.18, pp. 60/61] only for the functions h^l_M, h^l_F themselves. But it follows immediately from the proofs (and multiresolution arguments) that this applies also to the derivatives of these functions resulting in (2.321) (excluding $2x \in \mathbb{Z}$ if $k = l$). The cancellations (2.322) with $k = 0, \ldots, l-1$ are well-known properties for arbitrary wavelets generating an orthonormal basis in $L_2(\mathbb{R})$, having derivatives up to order $l-1$ and decaying strongly enough. However checking the corresponding proof in [Woj97, Proposition 3.1, pp. 47/48] it follows that in the above peculiar situation (where the wavelets are polynomials of degree l in each interval $(m, m+\frac{1}{2})$, $2m \in \mathbb{Z}$) this cancellation remains valid also for $k = l$. But this is well known and may also be found in [Bou95, p. 502] in case of homogeneous wavelets of the above type.

Remark 2.45. If $l = 0$ then (2.324) can be identified with the L_2-normalised version of the Haar basis in (2.96). Functions with the properties (i)–(iv) are called *splines*. The corresponding theory originates from the seminal work of Z. Ciesielski and his co-workers, preferably on the unit interval. We refer to [Cie75], [Cie79], [CiD72], [Rop76]. The connection with multiresolution analysis goes back to [Bat87], [Lem88]. These *spline wavelets* are also discussed in standard books on wavelets, [Dau92], [Mey92], [Woj97], [Mal99]. Spline functions have also been used to construct bases in some Besov spaces. We refer to [Rop76], [Bou95], [SiU08]. One may also consult [Tri81] and [T83, Section 2.12.3, pp. 184–187] where one finds further references.

First we ask what can be said about spline wavelets in the spaces $B^s_{pq}(\mathbb{R})$ and $F^s_{pq}(\mathbb{R})$ relying on the general theory of orthogonal wavelets as described in Section 1.1. Let b^-_{pq} and f^-_{pq} be the sequence spaces as introduced in (2.100)–(2.102). Representability and local convergence must be understood as before, for example as in connection with Theorem 1.18.

Theorem 2.46. *Let $l \in \mathbb{N}$ and let*

$$\left\{h^l_{jm} : j \in \mathbb{N}_{-1},\ m \in \mathbb{Z}\right\} \tag{2.325}$$

be an (L_∞-normalised) orthogonal spline basis in $L_2(\mathbb{R})$ according to Proposition 2.43.

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$ and*

$$\max\left(\frac{1}{p}, 1\right) - 1 - l < s < l. \tag{2.326}$$

Let $f \in S'(\mathbb{R})$. Then $f \in B^s_{pq}(\mathbb{R})$ if, and only if, it can be represented as

$$f = \sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} \mu_{jm}\, 2^{-j(s-\frac{1}{p})}\, h^l_{jm}, \quad \mu \in b^-_{pq}, \tag{2.327}$$

unconditional convergence being in $S'(\mathbb{R})$ *and locally in any space* $B^{\sigma}_{pq}(\mathbb{R})$ *with* $\sigma < s$. *The representation* (2.327) *is unique,*

$$\mu_{jm} = \mu_{jm}(f) = 2^{j(s-\frac{1}{p}+1)} \int_{\mathbb{R}} f(x)\, h^l_{jm}(x)\, \mathrm{d}x, \quad j \in \mathbb{N}_{-1},\ m \in \mathbb{Z}, \tag{2.328}$$

and

$$J:\ f \mapsto \mu(f) \tag{2.329}$$

is an isomorphic map of $B^s_{pq}(\mathbb{R})$ *onto* b^{-}_{pq}. *If, in addition,* $p < \infty$, $q < \infty$ *then*

$$\{2^{-j(s-\frac{1}{p})} h^l_{jm} : j \in \mathbb{N}_{-1},\ m \in \mathbb{Z}\} \tag{2.330}$$

is an unconditional (normalised) basis in $B^s_{pq}(\mathbb{R})$.

(ii) *Let* $0 < p < \infty$, $0 < q \le \infty$ *and*

$$\max\left(\frac{1}{p}, \frac{1}{q}, 1\right) - 1 - l < s < l. \tag{2.331}$$

Let $f \in S'(\mathbb{R})$. *Then* $f \in F^s_{pq}(\mathbb{R})$ *if, and only if, it can be represented as*

$$f = \sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} \mu_{jm}\, 2^{-j(s-\frac{1}{p})}\, h^l_{jm}, \quad \mu \in f^{-}_{pq}, \tag{2.332}$$

unconditional convergence being in $S'(\mathbb{R})$ *and locally in any space* $F^{\sigma}_{pq}(\mathbb{R})$ *with* $\sigma < s$. *The representation* (2.332) *is unique with* (2.328). *Furthermore,* J *in* (2.329) *is an isomorphic map of* $F^s_{pq}(\mathbb{R})$ *onto* f^{-}_{pq}. *If, in addition,* $q < \infty$ *then* (2.330) *is an unconditional (normalised) basis in* $F^s_{pq}(\mathbb{R})$.

Proof. This follows from a modification of the proof of Theorem 1.18 and Remark 1.19. We add a few comments. With σ_p and σ_{pq} as in (1.35) the conditions (1.69), (1.73) with $u = l$ coincide with (2.326), (2.331). Otherwise (1.70), (1.74) with $n = 1$ is the same as (2.327), (2.332) where we now adapted the normalisations to Theorem 2.9. Two additional comments are necessary. Beyond any technicalities (such as convergence, use of duality) the proof of Theorem 1.18 is based on atoms according to Theorem 1.7 with $K = L = u = l$ in (1.36), (1.39) and (1.34) in place of (1.31), (1.32) on the one hand, and local means according to Theorem 1.15 with $A = B = u = l$ in (1.50), (1.52) and (1.54) in place of (1.42), (1.43) on the other hand. But the weaker version as stated there with Lipschitz conditions for the highest derivatives involved is just what one needs now identifying splines of order l with atoms and kernels of local means. There is a second point. In contrast to wavelets in Theorem 1.18 and the underlying atoms and kernels of local means the splines h^l_{jm} do not have compact supports located near $2^{-j}m$. But they decay exponentially. This is sufficient for all estimates. Then one can prove the above theorem in the same way as Theorem 1.18 based on the references in Remark 1.19 and the Theorems 1.7, 1.15, which we streamlined just for this purpose. □

Remark 2.47. The above theorem complements the one-dimensional case of Theorem 1.18. There is no doubt that there is an n-dimensional counterpart. For this purpose one extends (2.176) to the above splines of order l,

$$h^{l,G}_{jm}(x) = \prod_{r=1}^{n} h^l_{G_r}(2^j x_r - m_r), \quad j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n, \tag{2.333}$$

with h^l_F, h^l_M as above, (2.322) in place of (2.172) and otherwise as in (2.173)–(2.176). Then one obtains the n-dimensional version of the above theorem by the same arguments as in the proof. This complements Theorem 1.18 now in its n-dimensional version. Similarly one can complement the wavelet representation in Theorem 1.54 for the spaces $S^r_{pq}B(\mathbb{R}^2)$ and $S^r_{pq}F(\mathbb{R}^2)$. For this purpose one modifies (1.198) by

$$h^l_{km}(x) = h^l_{k_1m_1}(x_1)\, h^l_{k_2m_2}(x_2), \quad k \in \mathbb{N}^2_{-1},\ m \in \mathbb{Z}^2. \tag{2.334}$$

Then one obtains Theorem 1.54 with h^l_{km} in place of ψ_{km} and the same restrictions of $p, q, s = r$ as in Theorem 2.46. There might be some interest in these spline representations for isotropic spaces $A^s_{pq}(\mathbb{R}^n)$ and spaces $S^r_{pq}A(\mathbb{R}^n)$ with dominating mixed smoothness, but *it is not what we are looking for*. In case of Haar functions, hence $l = 0$, the conditions (2.326), (2.331) would be empty, whereas we have Theorems 2.9, 2.21, 2.32, 2.38 which are sharp at least for the B-spaces. It is a challenging task to raise the theory of spline wavelets of order $l \in \mathbb{N}$ at the same level. In what follows we give partial answers.

2.5.2 Spline bases on $\mathbb{R}$

We ask for extensions of Theorem 2.9 from Haar bases (2.324) with $l = 0$ to corresponding spline bases with $l \in \mathbb{N}$. There are satisfactory assertions if $1 \le p < \infty$, $0 < q \le \infty$ in case of B-spaces and if $1 < p, q < \infty$ in case of F-spaces and some (less convincing) results for p, q otherwise. But we are not interested in most general formulations. We compromise and incorporate only handsome assertions for spaces with $p < 1$. First we extend the crucial Proposition 2.6 to spline systems (2.323) with the properties (i)–(iv) listed there. We assume $1 \le q < \infty$ in case of the F-spaces. Then (2.47) reduces to

$$\begin{cases} 0 < s < \frac{1}{p} & \text{if } q \le p < \infty, \\ \max\left(\frac{1}{p}, 1\right) - 1 < s < \frac{1}{q} & \text{if } \frac{1}{q} < \frac{1}{p} < 1 + \frac{1}{q}. \end{cases} \tag{2.335}$$

One may consult Figure 2.3, p. 82, for $H^s_p(\mathbb{R}) = F^s_{p,2}(\mathbb{R})$, $s > 0$. In case of $F^s_{pq}(\mathbb{R})$, $1 \le q < \infty$, $s > 0$, the breaking point $1/2$ must be replaced by $1/q$ and $3/2$ becomes $1 + \frac{1}{q}$. Let b^-_{pq} and f^-_{pq} be the same sequence spaces as in (2.100)–(2.102).

Proposition 2.48. *Let $l \in \mathbb{N}$ and let $\{h^l_{jm}\}$ be an orthogonal spline basis of order l according to Proposition 2.43.*

(i) *Let $0 < p < \infty$, $0 < q \le \infty$ and*

$$\max\left(\frac{1}{p}, 1\right) - 1 < s < l + \frac{1}{p}. \tag{2.336}$$

Then

$$f = \sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} \mu_{jm}\, 2^{-j(s-\frac{1}{p})}\, h^l_{jm}, \quad \mu \in b^-_{pq}, \tag{2.337}$$

belongs to $B^s_{pq}(\mathbb{R})$ and

$$\| f \,|B^s_{pq}(\mathbb{R})\| \le c\, \|\mu\,|b^-_{pq}\| \tag{2.338}$$

for some $c > 0$ and all $\mu \in b^-_{pq}$.

(ii) *Let $0 < p < \infty$, $1 \le q < \infty$ and*

$$\begin{cases} 0 < s < l + \frac{1}{p} & \text{if } q \le p, \\ \max\left(\frac{1}{p}, 1\right) - 1 < s < l + \frac{1}{q} & \text{if } \frac{1}{q} < \frac{1}{p} < 1 + \frac{1}{q}. \end{cases} \tag{2.339}$$

Then

$$f = \sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} \mu_{jm}\, 2^{-j(s-\frac{1}{p})}\, h^l_{jm}, \quad \mu \in f^-_{pq}, \tag{2.340}$$

belongs to $F^s_{pq}(\mathbb{R})$ and

$$\| f \,|F^s_{pq}(\mathbb{R})\| \le c\, \|\mu\,|f^-_{pq}\| \tag{2.341}$$

for some $c > 0$ and all $\mu \in f^-_{pq}$.

Proof. *Step* 1. We prove part (i). Recall that

$$\| f \,|B^s_{pq}(\mathbb{R})\| \sim \sum_{r=0}^{l} \| f^{(r)} \,|B^{s-l}_{pq}(\mathbb{R})\| \tag{2.342}$$

are equivalent quasi-norms, [T83, Theorem 2.3.8, pp. 58/59]. Let

$$\max\left(\frac{1}{p}, 1\right) - 1 < s - l < \frac{1}{p}. \tag{2.343}$$

Let temporarily $h^{l,r}_F(x) = (h^l_F)^{(r)}(x)$ and $h^{l,r}_M(x) = (h^l_M)^{(r)}(x)$ with $r = 0, \ldots, l$, and let $h^{l,r}_{jm}$ be the counterpart of (2.323). It follows from (2.337) that

$$f^{(r)} = \sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} \mu_{jm}\, 2^{-j(l-r)}\, 2^{-j(s-l-\frac{1}{p})}\, h^{l,r}_{jm}. \tag{2.344}$$

As mentioned in Remark 2.7 one can replace χ_{jm} in (2.45) by $h^{l,r}_{jm}$ using the decay properties (2.321). Then it follows from Proposition 2.6 (i) with $s-l$ in place of s that

$$\|f\,|B^s_{pq}(\mathbb{R})\| \sim \sum_{r=0}^{l} \|f^{(r)}\,|B^{s-l}_{pq}(\mathbb{R})\| \le c\,\|\mu\,|b^-_{pq}\|. \tag{2.345}$$

This proves (2.338) if s is restricted by (2.343). One obtains (2.338) by the same arguments with s as in (2.44). Application of real interpolation

$$\big(B^{s_0}_{pq}(\mathbb{R}), B^{s_1}_{pq}(\mathbb{R})\big)_{\theta,q} = B^s_{pq}(\mathbb{R}) \tag{2.346}$$

$0<\theta<1$, $s_0<s_1$, $s=(1-\theta)s_0+\theta s_1$, Theorem 1.22, gives the desired result.

Step 2. The proof of part (ii) is similar. There is a counterpart of (2.342) with F in place of B, [T83, Theorem 2.3.8, pp. 58/59]. Instead of (2.343) one uses now (2.335) with $s-l$ in place s. Then one can apply Proposition 2.6 (ii), Remark 2.7 and one can argue as above now based on the complex interpolation (1.83). □

After Proposition 2.48 has been established one can now extend Theorem 2.9 from Haar bases (which are spline bases of order $l=0$) to spline bases of order $l\in\mathbb{N}$. One can follow the proof of this theorem. The result will be stated below. But this might also be considered as a complement of Theorem 2.46. Comparing these results it comes out that the new approach based on Proposition 2.48 provides a decisive improvement of Theorem 2.46 in the expected way if $p\ge 1$. However if $p<1$ then the situation is different. There are assertions obtained now which are not covered by Theorem 2.46 (some of them will be formulated). On the other hand even if one applies interpolation to the results obtained by the new method there remain cases covered by Theorem 2.46 which cannot be reached in this way. One can interpolate all these spaces for which one has spline bases of order l. But we doubt that one obtains in this way final or natural assertions for spline bases of order l in spaces $B^s_{pq}(\mathbb{R})$ and $F^s_{pq}(\mathbb{R})$ with $p<1$. This may justify our restriction to direct counterparts of Theorem 2.9 which may be considered as a complement to Theorem 2.46. About the technicalities we refer to these two theorems and their proofs. This will not be repeated here. Let b^-_{pq} and f^-_{pq} be the sequence spaces according to (2.100)–(2.102).

Theorem 2.49. *Let $l\in\mathbb{N}$ and let*

$$\{h^l_{jm} : j\in\mathbb{N}_{-1},\ m\in\mathbb{Z}\} \tag{2.347}$$

be an (L_∞-normalised) orthogonal spline basis in $L_2(\mathbb{R})$ according to Proposition 2.43.

(i) *Theorem 2.46* (i) *remains valid for $0<p\le\infty$, $0<q\le\infty$ and*

$$\frac{1}{p}-1-l<s<l+\min\Big(\frac{1}{p},1\Big), \tag{2.348}$$

region within the broken lines in Figure 3.3, p. 170.

(ii) *Theorem* 2.46 (ii) *remains valid for* $0 < p < \infty$, $1 < q < \infty$ *and*

$$\begin{cases} \frac{1}{q} - 1 - l < s < \frac{1}{p} + l & \text{if } p \geq q, \\ \frac{1}{p} - 1 - l < s < \frac{1}{q} + l & \text{if } 1 < p < q, \\ \frac{1}{p} - 1 < s < \frac{1}{q} + l & \text{if } 1 \leq \frac{1}{p} < 1 + \frac{1}{q}. \end{cases} \tag{2.349}$$

Proof. Step 1. We follow Step 2 of the proof of Theorem 2.9. Let p, q, s be as in Proposition 2.48 (ii). If f is given by (2.340) = (2.332) then it follows $f \in F^s_{pq}(\mathbb{R})$ with (2.341). Conversely let $f \in F^s_{pq}(\mathbb{R})$ still with p, q, s as in Proposition 2.48 (ii) and let $\mu_{jm}(f)$ be the local means according to (2.328). By (2.322) the splines h^l_{jm} can be taken as kernels according to Definition 1.9 with $A = 0$ and $B = l + 1$ (different normalisations). Then it follows from Theorem 1.15 and $s < l + 1$ that $\mu(f) \in f^-_{pq}$ and

$$\| \mu(f) \,|\, f^-_{pq} \| \leq c \, \| f \,|\, F^s_{pq}(\mathbb{R}) \|. \tag{2.350}$$

But the rest is now the same as in Step 2 of the proof of Theorem 2.9. This proves part (ii) of the theorem for p, q, s as in Proposition 2.48 (ii). Afterwards one can use duality and interpolation in the same way as in Step 3 of the proof of Theorem 2.9. This covers all cases in (2.349).

Step 2. We prove part (i) of the theorem. Let $1 \leq p < \infty$. We rely now on part (i) of Proposition 2.48, argue as above and use in addition the interpolation (2.118). Afterwards it follows by complex interpolation according to (1.82) that the inequality (2.338) remains valid for $0 < q < \infty$ and

$$\frac{1}{p} - 1 - l < s < l + \frac{1}{p}, \quad 0 < p < \infty.$$

The splines h^l_{jm} are kernels according to Definition 1.9 with $A = l$, $B = l + 1$ and (1.50) coincides with (2.348) if $p < 1$. The rest is now the same as above using again in addition the interpolation (2.118). The spaces with $p = \infty$ can be incorporated as duals of spaces with $p = 1$ and interpolation. □

Remark 2.50. Part (i) is a satisfactory assertion and (2.348) is the natural generalisation of (2.107). One may also consult Proposition 2.24 and the optimality discussion about (homogeneous) spline wavelets in [Bou95, p. 503] if $p, q \geq 1$. Complex interpolation according to Theorem 1.22 can also be used in case of F-spaces. But in contrast to B-spaces the outcome does not look very natural. This may explain why we restricted $p < 1$ to the third line in (2.349) as it comes out without the indicated complex interpolation.

Remark 2.51. Let

$$H^s_p(\mathbb{R}) = F^s_{p,2}(\mathbb{R}), \quad 0 < p < \infty, \; s \in \mathbb{R}, \tag{2.351}$$

be the (fractional Hardy–)Sobolev spaces according to (1.17)–(1.19), extended to $p \leq 1$. Let

$$\{ h^l_{jm} : j \in \mathbb{N}_{-1}, \; m \in \mathbb{Z} \}, \quad l \in \mathbb{N}_0, \tag{2.352}$$

be an (L_∞-normalised) orthogonal spline basis in $L_2(\mathbb{R})$ according to Proposition 2.43 (including the Haar system $\{h_{jm}\} = \{h^0_{jm}\}$). Let

$$\begin{cases} -l - \frac{1}{2} < s < l + \frac{1}{p} & \text{if } p \ge 2, \\ -l - 1 + \frac{1}{p} < s < l + \frac{1}{2} & \text{if } 1 < p < 2, \\ \frac{1}{p} - 1 < s < l + \frac{1}{2} & \text{if } \frac{2}{3} < p \le 1. \end{cases} \tag{2.353}$$

Then $\{h^l_{jm}\}$ is an unconditional basis in $H^s_p(\mathbb{R})$ and

$$\|f \,|H^s_p(\mathbb{R})\| \sim \Big\| \Big(\sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} |\mu_{jm}(f)\, \chi^{(p)}_{jm}(\cdot)|^2 \Big)^{1/2} |L_p(\mathbb{R}) \Big\| \tag{2.354}$$

with $\mu_{jm}(f)$ as in (2.328) (equivalent quasi-norms). This follows from part (ii) of Theorem 2.49. It extends Remark 2.12 from $l = 0$ (Haar system) to $l \in \mathbb{N}_0$. If $l \in \mathbb{N}$ and $p \le 1$ then it follows from Theorem 2.46 and Remark 2.50 that (2.354) remains valid also for some other spaces $H^s_p(\mathbb{R})$.

2.5.3 Spline wavelet bases on $\mathbb{R}^n$

Let $\{2^{j/2} h^l_{jm}\}$, $l \in \mathbb{N}_0$, be an orthonormal spline basis in $L_2(\mathbb{R})$ according to Proposition 2.43 with the underlying scaling function h^l_F and wavelet h^l_M. Let G be as in (2.173), (2.174) where $n \in \mathbb{N}$. Then

$$\{h^{l,G}_{jm} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n\}, \quad l \in \mathbb{N}_0, \tag{2.355}$$

with

$$h^{l,G}_{jm}(x) = \prod_{r=1}^{n} h^l_{G_r}\big(2^j x_r - m_r\big), \quad j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n, \tag{2.356}$$

is the generalisation of the Haar wavelet basis $\{h^G_{jm}\} = \{h^{0,G}_{jm}\}$ in (2.175) to orthogonal spline wavelet bases of order $l \in \mathbb{N}_0$ in $L_2(\mathbb{R}^n)$. But otherwise we are in the same position as in Sections 2.3.1, 2.3.2. This applies also to the n-dimensional extension of Theorem 2.46 based on Theorem 1.18. This will not be formulated here. Furthermore as discussed in Remark 2.50 that the assertions obtained so far for spaces on $\mathbb{R}$ with $p < 1$ are not really satisfactory in case of F-spaces. This is even worse if one steps from one dimension to higher dimensions now both for B-spaces and F-spaces. This may justify that we restrict the formulation below to $p > 1$ (and in case of F-spaces $q > 1$) which are anyway the most interesting cases at least in this context. Let $b_{pq}(\mathbb{R}^n)$ and $f_{pq}(\mathbb{R}^n)$ be the sequence spaces introduced in Definition 2.18.

Theorem 2.52. *Let $n \in \mathbb{N}$ and $l \in \mathbb{N}_0$. Let $\{h^{l,G}_{jm}\}$ be the above orthogonal spline basis in $L_2(\mathbb{R}^n)$ of order l.*

(i) *Let $1 < p \le \infty$, $0 < q \le \infty$, and*

$$\frac{1}{p} - 1 - l < s < \frac{1}{p} + l. \tag{2.357}$$

Let $f \in S'(\mathbb{R}^n)$. Then $f \in B^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \mu^{l,G}_{jm}\, 2^{-j(s-\frac{n}{p})}\, h^{l,G}_{jm}, \quad \mu \in b_{pq}(\mathbb{R}^n), \tag{2.358}$$

unconditional convergence being in $S'(\mathbb{R}^n)$ and locally in any space $B^\sigma_{pq}(\mathbb{R}^n)$ with $\sigma < s$. The representation (2.358) is unique,

$$\mu^{l,G}_{jm} = \mu^{l,G}_{jm}(f) = 2^{j(s-\frac{n}{p}+n)} \int_{\mathbb{R}^n} f(x)\, h^{l,G}_{jm}(x)\, \mathrm{d}x, \quad j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n, \tag{2.359}$$

and

$$J : f \mapsto \mu(f) \tag{2.360}$$

is an isomorphic map of $B^s_{pq}(\mathbb{R}^n)$ onto $b_{pq}(\mathbb{R}^n)$. If, in addition, $p < \infty$, $q < \infty$, then

$$\left\{2^{-j(s-\frac{n}{p})}\, h^{l,G}_{jm} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n\right\} \tag{2.361}$$

is an unconditional (normalised) basis in $B^s_{pq}(\mathbb{R}^n)$.

(ii) *Let $1 < p < \infty$, $1 < q < \infty$, and*

$$\begin{cases} \frac{1}{q} - 1 - l < s < \frac{1}{p} + l & \text{if } p \ge q, \\ \frac{1}{p} - 1 - l < s < \frac{1}{q} + l & \text{if } p < q. \end{cases} \tag{2.362}$$

Let $f \in S'(\mathbb{R}^n)$. Then $f \in F^s_{pq}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{j=0}^{\infty} \sum_{G \in G^j} \sum_{m \in \mathbb{Z}^n} \mu^{l,G}_{jm}\, 2^{-j(s-\frac{n}{p})}\, h^{l,G}_{jm}, \quad \mu \in f_{pq}(\mathbb{R}^n), \tag{2.363}$$

unconditional convergence being in $F^s_{pq}(\mathbb{R}^n)$. The representation is unique with $\mu^{l,G}_{jm}(f)$ as in (2.359) and J in (2.360) is an isomorphic map of $F^s_{pq}(\mathbb{R}^n)$ onto $f_{pq}(\mathbb{R}^n)$. Furthermore, (2.361) is an unconditional (normalised) basis in $F^s_{pq}(\mathbb{R}^n)$.

Proof. This is the direct counterpart of Theorem 2.21. The proof and the related references are the same as there. We rely now on Theorem 2.49 with the indicated restrictions for p and q. □

Corollary 2.53. *Let $n \in \mathbb{N}$, $l \in \mathbb{N}_0$, $1 < p < \infty$, and*

$$\begin{cases} -l - \frac{1}{2} < s < l + \frac{1}{p} & \text{if } p \geq 2, \\ -l - 1 + \frac{1}{p} < s < l + \frac{1}{2} & \text{if } p < 2. \end{cases} \tag{2.364}$$

Then $\{2^{-j(s-\frac{n}{p})} h^{l,G}_{jm}\}$ according to (2.361) is an unconditional basis in the Sobolev spaces $H^s_p(\mathbb{R}^n)$ in Remark 1.2 (iii) *and*

$$\|f \,|H^s_p(\mathbb{R}^n)\| \sim \Big\| \Big(\sum_{j,G,m} |\mu^{l,G}_{jm}(f)\, \chi^{(p)}_{jm}(\cdot)|^2 \Big)^{1/2} |L_p(\mathbb{R}^n) \Big\| \tag{2.365}$$

with $\mu^{l,G}_{jm}(f)$ as in (2.359).

Proof. This is special case of Theorem 2.52 with $H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$. □

Remark 2.54. The above corollary extends Corollary 2.23 (with $p > 1$) from $l = 0$ (Haar functions) to $l \in \mathbb{N}_0$.

2.5.4 Spline tensor bases on $\mathbb{R}^2$

Sections 2.4.2, 2.4.3 dealt with Haar tensor bases in some spaces $S^r_p H(\mathbb{R}^2)$ and $S^r_{pq} B(\mathbb{R}^2)$ with dominating mixed smoothness. This can be extended partly to corresponding spline tensor bases of order $l \in \mathbb{N}$. Let $\{2^{j/2} h^l_{jm}\}$, $l \in \mathbb{N}_0$, be an orthonormal spline basis in $L_2(\mathbb{R})$ according to Proposition 2.43. Then

$$\{h^l_{km} : k \in \mathbb{N}^2_{-1},\ m \in \mathbb{Z}^2\} \tag{2.366}$$

with

$$h^l_{km}(x) = h^l_{k_1 m_1}(x_1)\, h^l_{k_2 m_2}(x_2), \quad k \in \mathbb{N}^2_{-1},\ m \in \mathbb{Z}^2,\ x \in \mathbb{R}^2, \tag{2.367}$$

is the extension of the Haar tensor basis (2.240), (2.241) to an (L_∞-normalised) orthogonal spline tensor basis of order $l \in \mathbb{N}_0$ in $L_2(\mathbb{R}^2)$. Otherwise we are largely in the same position as in case of Haar tensor bases. Let $s_{pq}b(\mathbb{R}^2)$ and $s_{pq}f(\mathbb{R}^2)$ be the sequence spaces as introduced in Definition 2.30 and let $S^r_{pq}B(\mathbb{R}^2)$ and $S^r_p H(\mathbb{R}^2)$ be the spaces with dominating mixed smoothness according to Definition 1.38 and (2.231)–(2.234). Then Theorems 2.32, 2.38 can be extended from Haar bases to spline bases as follows.

Theorem 2.55. *Let $l \in \mathbb{N}_0$ and let $\{h^l_{km}\}$ be the above orthogonal spline basis in $L_2(\mathbb{R}^2)$ of order l.*

(i) *Let $0 < p \leq \infty$, $0 < q \leq \infty$ ($1 < q \leq \infty$ if $p = \infty$) and*

$$\frac{1}{p} - 1 - l < r < l + \min\Big(\frac{1}{p}, 1\Big), \tag{2.368}$$

region within the broken lines in Figure 3.3, p. 170. Let $f \in S'(\mathbb{R}^2)$. *Then* $f \in S^r_{pq}B(\mathbb{R}^2)$ *if, and only if, it can be represented as*

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{Z}^2} \mu_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})}\, h^l_{km}, \quad \mu \in s_{pq}b(\mathbb{R}^2). \tag{2.369}$$

This representation is unique,

$$\mu_{km} = \mu_{km}(f) = 2^{(k_1+k_2)(r-\frac{1}{p}+1)} \int_{\mathbb{R}^2} f(x)\, h^l_{km}(x)\, \mathrm{d}x, \tag{2.370}$$

and

$$J: f \mapsto \{\mu_{km}(f)\} \tag{2.371}$$

is an isomorphic map of $S^r_{pq}B(\mathbb{R}^2)$ *onto* $s_{pq}b(\mathbb{R}^2)$. *If* $p < \infty$, $q < \infty$ *then* (2.366) *is an unconditional basis in* $S^r_{pq}B(\mathbb{R}^2)$.

(ii) *Let*

$$\begin{cases} -l - \frac{1}{2} < r < l + \frac{1}{p} & \text{if } 2 \le p < \infty, \\ -l - 1 + \frac{1}{p} < r < l + \frac{1}{2} & \text{if } 1 < p < 2. \end{cases} \tag{2.372}$$

Let $f \in S'(\mathbb{R}^2)$. *Then* $f \in S^r_p H(\mathbb{R}^2)$ *if, and only if, it can be represented as*

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{Z}^2} \mu_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})}\, h^l_{km}, \quad \mu \in s_{p,2}f(\mathbb{R}^2). \tag{2.373}$$

This representation is unique with (2.370), J *in* (2.371) *is an isomorphic map of* $S^r_p H(\mathbb{R}^2)$ *onto* $s_{p,2}f(\mathbb{R}^2)$, *and* (2.366) *is an unconditional basis in* $S^r_p H(\mathbb{R}^2)$.

Proof. Step 1. We prove part (ii). By Theorem 2.49 (ii) and Remark 2.51 the L_2-orthogonal spline system $\{h^l_{jm}\}$ in (2.347) is a basis in $H^r_p(\mathbb{R})$ for p, r as in (2.372). Then it follows again from [SiU08, Corollary 2.6] that $\{h^l_{km}\}$ in (2.372) is a basis $S^r_p H(\mathbb{R}^2)$ with the indicated properties.

Step 2. We prove part (i). The case $l = 0$ (Haar system) is covered by Theorem 2.38. Let $l \in \mathbb{N}$. Recall that

$$\| f \,| S^r_{pq}B(\mathbb{R}^2)\| \sim \sum_{\substack{0 \le \alpha_1 \le l \\ 0 \le \alpha_2 \le l}} \| D^\alpha f \,| S^{r-l}_{pq}B(\mathbb{R}^2)\|, \tag{2.374}$$

[ST87, Theorem 2, Section 2.2.6, pp. 98/99]. Let

$$\max\left(\frac{1}{p}, 1\right) - 1 < r - l < \frac{1}{p} \tag{2.375}$$

and let f be given by (2.369). With the same abbreviation on $\mathbb{R}$ as used in (2.344) we complement (2.367) by

$$h^{l,\alpha}_{km}(x) = h^{l,\alpha_1}_{k_1 m_1}(x_1)\, h^{l,\alpha_2}_{k_2 m_2}(x_2), \quad k \in \mathbb{N}^2_{-1},\ m \in \mathbb{Z}^2,\ x \in \mathbb{R}^2, \tag{2.376}$$

where $\alpha = (\alpha_1, \alpha_2)$ with $0 \le \alpha_1, \alpha_2 \le l$. Then it follows from (2.369) that

$$D^\alpha f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{Z}^2} \mu^\alpha_{km} \, 2^{-(k_1+k_2)(r-l-\frac{1}{p})} \, h^{l,\alpha}_{km} \tag{2.377}$$

with

$$\mu^\alpha_{km} = 2^{-k_1(l-\alpha_1)-k_2(l-\alpha_2)} \, \mu_{km}. \tag{2.378}$$

We apply Proposition 2.34, modified according to Remark 2.36, with $r - l$ in place of r. Then it follows from (2.374) that $f \in S^r_{pq}B(\mathbb{R}^2)$ and

$$\| f \, | S^r_{pq}B(\mathbb{R}^2)\| \le c \, \|\mu \, | s_{pq}b(\mathbb{R}^2)\|. \tag{2.379}$$

One obtains (2.379) by the same arguments if r is restricted by (2.249). The real interpolation (2.274) applies to $0 < p < \infty$ and $q_0 = q_1 = q$. This follows from (2.277) with the same references as there. Then one obtains by interpolation of the two described cases that (2.379) remains valid for all

$$\max\left(\frac{1}{p}, 1\right) - 1 < r < l + \min\left(\frac{1}{p}, 1\right), \quad 0 < p < \infty, \; 0 < q < \infty. \tag{2.380}$$

Conversely let $f \in S^r_{pq}B(\mathbb{R}^2)$. Then it follows from Theorem 1.52 and Remark 1.53 with $A = 0$ and $B = l + 1$ that $\mu \in s_{pq}b(\mathbb{R}^2)$. In other words, if f can be represented by (2.369), (2.370) then one obtains

$$\| f \, | S^r_{pq}B(\mathbb{R}^2)\| \sim \|\mu(f) \, | s_{pq}b(\mathbb{R}^2)\| \tag{2.381}$$

where p, q, r are restricted by (2.380). For $p \ge 1$ one can follow the arguments in the proof of Theorem 2.38. To extend this assertion to $p < 1$ we first remark that there is a counterpart of the complex interpolation (1.82) with $S^r_{pq}B(\mathbb{R}^2)$ in place of $B^s_{pq}(\mathbb{R}^2)$, [Vyb06, Theorem 4.6]. Then one can argue as in Step 2 of the proof of Theorem 2.49. □

Remark 2.56. Theorem 1.54 gives a satisfactory answers for general wavelet representations of spaces with dominating mixed smoothness. We discussed in Remark 2.47 what happens in the context of general wavelet theory if one deals with h^l_{km} in (2.367) = (2.334) in place of ψ_{km} in (1.198). Then one has again Theorem 1.54 with $u = l$ or an extension of Theorem 2.46 from h^l_{jm} in $\mathbb{R}$ to h^l_{km} in $\mathbb{R}^2$ with the same restrictions for the parameters. It is one of the main aims of Section 2.5 to find out which improvements can be expected if one takes into account the very specific nature of spline tensor systems. In case of the Haar tensor system (spline tensor system of order $l = 0$) the answers given in Theorems 2.32, 2.38 are more or less satisfactory. In case of spline tensor systems of order $l \in \mathbb{N}$ one has by Theorem 2.55 at least for the B-spaces convincing assertions. But the situation is different for the H-spaces and the F-spaces, especially if $p < 1$, similarly as for spline systems on $\mathbb{R}$ and spline wavelet systems, Remark 2.50, Section 2.5.3. This is also the reason why we did not try to improve the assertions of the above theorem somewhat by interpolation. There is little hope to obtain something natural in this way.

Chapter 3
Faber bases

3.1 Faber bases on intervals

3.1.1 Introduction and preliminaries

Let $I = (0, 1)$ be the unit interval on $\mathbb{R}$. Recall that the Faber system on I,

$$\big\{ v_0,\, v_1,\, v_{jm} \,:\, j \in \mathbb{N}_0;\; m = 0, \ldots, 2^j - 1 \big\} \tag{3.1}$$

consists of the basic functions

$$v_0(x) = 1 - x, \quad v_1(x) = x \quad \text{where } 0 \le x \le 1, \tag{3.2}$$

and the hat-functions $v_{jm}(x)$, $0 \le x \le 1$,

$$v_{jm}(x) = \begin{cases} 2^{j+1}\big(x - 2^{-j}m\big) & \text{if } 2^{-j}m \le x < 2^{-j}m + 2^{-j-1}, \\ 2^{j+1}\big(2^{-j}(m+1) - x\big) & \text{if } 2^{-j}m + 2^{-j-1} \le x < 2^{-j}(m+1), \\ 0 & \text{otherwise}, \end{cases} \tag{3.3}$$

(2.3)–(2.5), Figure 2.1, p. 64. Recall that $C(I)$ is the space of all complex-valued continuous functions on the closed interval $\bar{I} = [0, 1]$, furnished with the L_∞-norm. According to Theorem 2.1 (iii) the Faber system (3.1)–(3.3) is a basis in $C(I)$. Any $f \in C(I)$ can be represented by

$$f(x) = f(0)\, v_0(x) + f(1)\, v_1(x) - \frac{1}{2} \sum_{j=0}^{\infty} \sum_{m=0}^{2^j - 1} \big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m)\, v_{jm}(x), \tag{3.4}$$

$0 \le x \le 1$, where

$$-\tfrac{1}{2}\big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m) = f(2^{-j}m + 2^{-j-1}) - \tfrac{1}{2} f(2^{-j}m) - \tfrac{1}{2} f(2^{-j}m + 2^{-j}) \tag{3.5}$$

are the same second differences as in (2.7). This is Faber's observation, [Fab09], 1909, and Step 4 of the proof of Theorem 2.1 follows essentially his arguments. Historical comments may be found in Remark 2.3. The second distinguished set of functions on I at that time is the Haar system in (2.1), (2.2), [Haar10], 1910. We collected (and proved) in Theorem 2.1 some related assertions and added in Remarks 2.3, 2.4 historical comments. Chapter 2 dealt with Haar bases in some spaces of type B^s_{pq} and F^s_{pq}. One may ask for counterparts with Faber bases in place of Haar bases. However there are some crucial differences. The multiresolution property of the orthogonal Haar system $\{h_0, h_{jm}\}$ according to (2.1) gives the possibility to step from $A^s_{pq}(\mathbb{R})$ and $A^s_{pq}(I)$ to corresponding isotropic spaces $A^s_{pq}(\mathbb{R}^n)$ and $A^s_{pq}(\mathbb{Q}^n)$ in higher dimensions. This has been done in Section 2.3 where the resulting systems are called *Haar wavelet bases*. There is no counterpart of this method in terms of Faber systems. Section 2.4 dealt with *Haar tensor bases* in spaces $S^r_{pq}A(\mathbb{R}^n)$ and $S^r_{pq}A(\mathbb{Q}^n)$ with dominating mixed smoothness. The underlying procedure admits to replace Haar bases on I by Faber bases on I. Then we obtain *Faber tensor bases* in higher dimensions which we call simply *Faber bases* (because there are no notationally competing Faber wavelet bases). The study of these Faber bases in spaces $S^r_{pq}A(\mathbb{Q}^n)$ with dominating mixed smoothness is the main topic of this Chapter 3. But first we ask whether the above Faber system in $C(I)$ is also a basis in some spaces $A^s_{pq}(I)$.

Recall that $A^s_{pq}(I)$ with $A = B$ or $A = F$ is the restriction of $A^s_{pq}(\mathbb{R})$ to I according to Definition 1.24 (i). One has for all $0 < p, q \le \infty$ ($p < \infty$ in case of F-spaces) and $s \in \mathbb{R}$ that

$$A^s_{pq}(I) = \left\{ f \in A^{s-1}_{pq}(I) : f' \in A^{s-1}_{pq}(I) \right\} \tag{3.6}$$

and

$$\| f \,|A^s_{pq}(I)\| \sim \| f \,|A^{s-1}_{pq}(I)\| + \| f' \,|A^{s-1}_{pq}(I)\| \tag{3.7}$$

(equivalent quasi-norms). This well-known assertion is a special case of [T08, Proposition 4.21, p. 113]. By (3.4) Faber bases in $A^s_{pq}(I)$ can only be expected if $A^s_{pq}(I)$ is continuously embedded in $C(I)$. Recall that

$$B^s_{pq}(I) \hookrightarrow C(I) \quad \text{if, and only if,} \quad \begin{cases} 0 < p \le \infty,\ 0 < q \le \infty, & s > \frac{1}{p}, \\ 0 < p \le \infty,\ 0 < q \le 1, & s = \frac{1}{p}, \end{cases} \tag{3.8}$$

and

$$F^s_{pq}(I) \hookrightarrow C(I) \quad \text{if, and only if,} \quad \begin{cases} 0 < p < \infty,\ 0 < q \le \infty, & s > \frac{1}{p}, \\ 0 < p \le 1,\ 0 < q \le \infty, & s = \frac{1}{p}. \end{cases} \tag{3.9}$$

We refer to [T08, Section 6.4.6, p. 229] where one finds the n-dimensional version of (3.8), (3.9). But the assertion itself is well known. One may consult [T01, Section 11] and [Har07, Section 7.2]. The sharp limiting cases in $\mathbb{R}^n$ go back to [SiT95]. We deal here with the non-limiting case

$$A^s_{pq}(I) \hookrightarrow C(I) \quad \text{if} \quad 0 < p, q \le \infty,\ s > \tfrac{1}{p}, \tag{3.10}$$

($p < \infty$ for F-spaces). Then

$$\| f \,|A^s_{pq}(I)\| \sim |f(0)| + |f(1)| + \| f' \,|A^{s-1}_{pq}(I)\| \tag{3.11}$$

are equivalent quasi-norms. Although assertions of this type are well known we add a short proof. By (3.7) and (3.10) it is sufficient to show that there is a constant $c > 0$ such that

$$\| f \,|A^{s-1}_{pq}(I)\| \le c\,|f(0)| + c\,|f(1)| + c\,\| f' \,|A^{s-1}_{pq}(I)\| \tag{3.12}$$

for all $f \in A^s_{pq}(I)$. We assume that there is no constant $c > 0$ with (3.12). Then one finds for any $c = j \in \mathbb{N}$ an element $f_j \in A^s_{pq}(I)$ with

$$\| f_j \,| A^{s-1}_{pq}(I)\| = 1 \quad \text{and} \quad |f_j(0)| + |f_j(1)| + \| f_j' \,| A^{s-1}_{pq}(I)\| \le j^{-1}. \tag{3.13}$$

Now it follows from (3.7) that the sequence $\{f_j\}$ is bounded in $A^s_{pq}(I)$ and hence pre-compact in $A^{s-1}_{pq}(I)$. We may assume that f_j converges in $A^{s-1}_{pq}(I)$ to some element f. By (3.13) and (3.7) this sequence converges even in $A^s_{pq}(I)$ to $f \in A^s_{pq}(I)$ and one obtains by (3.13),

$$\| f \,| A^{s-1}_{pq}(I)\| = 1, \quad f(0) = f(1) = 0, \quad f' = 0. \tag{3.14}$$

This is a contradiction.

3.1.2 Faber bases

We ask for the counterpart of Theorem 2.13 where we dealt with Haar bases in some spaces $A^s_{pq}(I)$. First we adapt the sequence spaces $b_{pq}(I)$ and $f_{pq}(I)$ according to (2.130)–(2.132) as follows. Let again $\chi^{(p)}_{jm}(x) = 2^{j/p}\chi_{jm}(x)$, where χ_{jm} is the characteristic function of the interval

$$I_{jm} = \big[2^{-j}m,\, 2^{-j}(m+1)\big), \quad j \in \mathbb{N}_0;\ m = 0, \ldots, 2^j - 1. \tag{3.15}$$

Let $b^+_{pq}(I)$ and $f^+_{pq}(I)$ with $0 < p, q \le \infty$ be the spaces of all sequences

$$\mu = \big\{\mu_0,\, \mu_1,\, \mu_{jm} : j \in \mathbb{N}_0;\ m = 0, \ldots, 2^j - 1\big\} \subset \mathbb{C} \tag{3.16}$$

quasi-normed by

$$\|\mu \,| b^+_{pq}(I)\| = |\mu_0| + |\mu_1| + \Big(\sum_{j=0}^{\infty} \Big(\sum_{m=0}^{2^j-1} |\mu_{jm}|^p \Big)^{q/p} \Big)^{1/q} \tag{3.17}$$

and

$$\|\mu \,| f^+_{pq}(I)\| = |\mu_0| + |\mu_1| + \Big\| \Big(\sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} |\mu_{jm}\, \chi^{(p)}_{jm}(\cdot)|^q \Big)^{1/q} |L_p(I)\Big\|, \tag{3.18}$$

usual modification if $p = \infty$ and /or $q = \infty$. Recall that one has (3.10) for $s > 1/p$.

Theorem 3.1. *Let $\{v_0, v_1, v_{jm}\}$ be the Faber system (3.1)–(3.3).*

(i) *Let $0 < p \le \infty$, $0 < q \le \infty$ and*

$$\frac{1}{p} < s < 1 + \min\Big(\frac{1}{p}, 1\Big), \tag{3.19}$$

Figure 3.1. Let $f \in D'(I)$. Then $f \in B^s_{pq}(I)$ if, and only if, it can be represented as

$$f = \mu_0 v_0 + \mu_1 v_1 + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} \mu_{jm} 2^{-j(s-\frac{1}{p})} v_{jm}, \quad \mu \in b^+_{pq}(I), \tag{3.20}$$

unconditional convergence being in $B^\sigma_{pq}(I)$ with $\sigma < s$ and in $C(I)$. The representation (3.20) is unique, $\mu = \mu(f)$, with

$$\mu_0(f) = f(0), \quad \mu_1(f) = f(1), \quad \mu_{jm}(f) = -2^{j(s-\frac{1}{p})-1} (\Delta^2_{2^{-j-1}} f)(2^{-j} m), \tag{3.21}$$

where $j \in \mathbb{N}_0$ and $m = 0, \ldots, 2^j - 1$. Furthermore,

$$J : f \mapsto \mu(f) \tag{3.22}$$

is an isomorphic map of $B^s_{pq}(I)$ onto $b^+_{pq}(I)$. If, in addition, $p < \infty$, $q < \infty$, then

$$\big\{ v_0, v_1, 2^{-j(s-\frac{1}{p})} v_{jm} : j \in \mathbb{N}_0;\ m = 0, \ldots, 2^j - 1 \big\} \tag{3.23}$$

is an unconditional (normalised) basis in $B^s_{pq}(I)$.

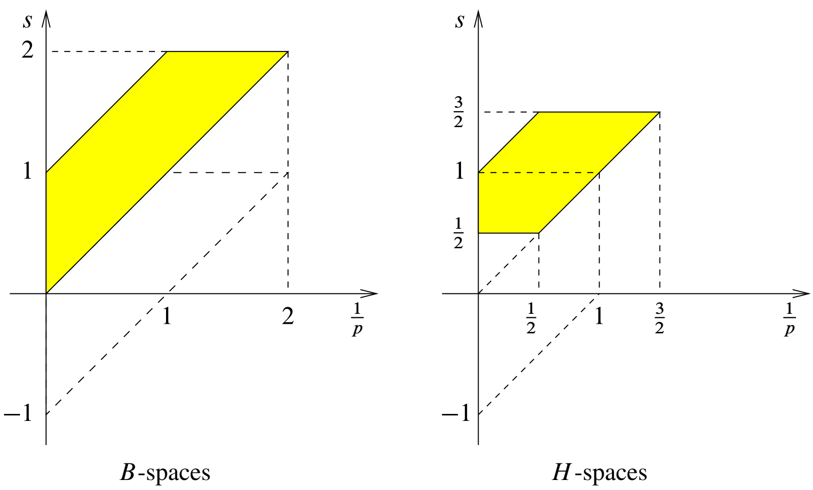

Figure 3.1. Faber bases.

(ii) *Let*

$$\begin{cases} 0 < p < \infty,\ 0 < q < \infty, & \max\big(\frac{1}{p}, \frac{1}{q}, 1\big) < s < 1 + \min\big(\frac{1}{p}, \frac{1}{q}, 1\big), \\ 1 < p < \infty,\ 1 < q < \infty, & s = 1, \\ 1 < p < \infty,\ 1 < q \le \infty, & \max\big(\frac{1}{p}, \frac{1}{q}\big) < s < 1. \end{cases} \tag{3.24}$$

Let $f \in D'(I)$. Then $f \in F^s_{pq}(I)$ if, and only if, it can be represented as

$$f = \mu_0 \, v_0 + \mu_1 \, v_1 + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} \mu_{jm} \, 2^{-j(s-\frac{1}{p})} v_{jm}, \quad \mu \in f^+_{pq}(I), \tag{3.25}$$

unconditional convergence being in $F^\sigma_{pq}(I)$ with $\sigma < s$ and in $C(I)$. The representation (3.25) is unique with $\mu_0(f)$, $\mu_1(f)$, $\mu_{jm}(f)$ as in (3.21) and J in (3.22) is an isomorphic map of $F^s_{pq}(I)$ onto $f^+_{pq}(I)$. If, in addition, $q < \infty$, then (3.23) is an unconditional (normalised) basis in $F^s_{pq}(I)$.

Proof. Step 1. Let f be given by (3.20) or (3.25) with $\mu \in a^+_{pq}(I)$ where $a \in \{b, f\}$. Then

$$f_j = \sum_{m=0}^{2^j-1} \mu_{jm} \, 2^{-j(s-\frac{1}{p})} \, v_{jm} \in C(I), \quad j \in \mathbb{N}_0, \tag{3.26}$$

with

$$\| f_j \, | C(I) \| = 2^{-j(s-\frac{1}{p})} \, \max_m |\mu_{jm}| \le 2^{-j(s-\frac{1}{p})} \, \| \mu \, | a^+_{pq}(I) \|. \tag{3.27}$$

Since $s > 1/p$ it follows that $f \in C(I)$ and

$$\| f \, | C(I) \| \le c \, \| \mu \, | a^+_{pq}(I) \|. \tag{3.28}$$

By (3.4) and Theorem 2.1 (iii) the representations (3.20), (3.25) are unique and one has (3.21).

Step 2. The Lipschitz functions v_{jm} are (not normalised) atoms according to Definition 1.5 with $K = 1$ and $L = 0$. Then it follows from Theorem 1.7 for f given by (3.20) or (3.25) that

$$f \in B^\sigma_{pq}(I) \quad \text{if} \quad \max\Big(\frac{1}{p}, 1\Big) - 1 < \sigma < \min(s, 1). \tag{3.29}$$

Then one has in any case covered by (3.19), (3.24) that

$$f \in A^{s-1}_{pq}(I) \quad \text{with} \quad \| f \, | A^{s-1}_{pq}(I) \| \le c \, \| \mu \, | a^+_{pq}(I) \|. \tag{3.30}$$

Step 3. Let $\{h_0, h_{jm}\}$ be the Haar system (2.128). Then it follows from (3.20), (3.25) that

$$f' = (\mu_1 - \mu_0) \, h_0 + 2 \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} \mu_{jm} \, 2^{-j(s-1-\frac{1}{p})} \, h_{jm}, \quad \mu \in a^+_{pq}(I). \tag{3.31}$$

The restrictions for p, q, s in the above theorem coincide with the restrictions in Theorem 2.13 for p, q and $s - 1$ in place of s. Then

$$f' \in A^{s-1}_{pq}(I), \quad \| f' \, | A^{s-1}_{pq}(I) \| \sim \| \mu \, | a^+_{pq}(I) \| - |\mu_1| - |\mu_0| + |\mu_1 - \mu_0|. \tag{3.32}$$

Now one obtains by (3.6), (3.7) and (3.30), (3.32) that

$$f \in A^s_{pq}(I) \quad \text{and} \quad \|f \,|A^s_{pq}(I)\| \le c\,\|a^+_{pq}(I)\|. \tag{3.33}$$

The converse

$$\|\mu\,|a^+_{pq}(I)\| \le c\,\|f\,|A^s_{pq}(I)\| \tag{3.34}$$

is a consequence of (3.32), (3.11) and $\mu_0 = f(0)$, $\mu_1 = f(1)$. This covers all assertions of the theorem, including that J in (3.22) is the indicated isomorphic map and that (3.23) is a (normalised) unconditional basis in $A^s_{pq}(I)$ if $p < \infty$, $q < \infty$. □

Remark 3.2. The above theorem applies to $A^s_{pq}(I)$ if, and only if, $A^{s-1}_{pq}(I)$ is covered by Theorem 2.13. Smoothness is lifted by 1 and Haar bases are replaced by Faber bases. One may compare the Figures 2.3, p. 82 and 3.1, p. 127.

Let again

$$H^s_p(\mathbb{R}) = F^s_{p,2}(\mathbb{R}), \quad 0 < p < \infty,\ s \in \mathbb{R}, \tag{3.35}$$

be the (fractional Hardy–)Sobolev spaces according to (1.17)–(1.19) extended to $p \le 1$, and let $H^s_p(I)$ be the restriction of $H^s_p(\mathbb{R})$ to I.

Corollary 3.3. *Let* $0 < p < \infty$ *and*

$$\begin{cases} 2 \le p < \infty, & \frac{1}{2} < s < 1 + \frac{1}{p}, \\ p < 2, & \frac{1}{p} < s < \frac{3}{2}, \end{cases} \tag{3.36}$$

Figure 3.1, p. 127. Then (3.23) is an unconditional basis in $H^s_p(I)$. Furthermore, $f \in D'(I)$ is an element of $H^s_p(I)$ if, and only if, it can be represented as

$$f = \mu_0\, v_0 + \mu_1\, v_1 + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} \mu_{jm}\, 2^{-j(s-\frac{1}{p})}\, v_{jm}, \quad \mu \in f^+_{p,2}(I). \tag{3.37}$$

The representation is unique with $\mu_0(f)$, $\mu_1(f)$, $\mu_{jm}(f)$ as in (3.21) and J in (3.22) is an isomorphic map of $H^s_p(I)$ onto $f^+_{p,2}(I)$.

Proof. This is a special case of Theorem 3.1 (ii) with $q = 2$ in (3.24). □

Remark 3.4. We refer to Remark 2.12 for the corresponding assertion in terms of Haar bases. One may again compare Figure 2.3, p. 82 with Figure 3.1, p. 127. The above corollary covers in particular the classical Sobolev spaces

$$W^1_p(I) = H^1_p(I), \quad 1 < p < \infty, \tag{3.38}$$

which will play a role later on.

3.1.3 Complements

For later use it will be helpful to reformulate and complement some assertions of Theorem 3.1. Let again $I = (0, 1)$ be the unit interval on $\mathbb{R}$, and let

$$\big(\Delta^M_{h,I} f\big)(x) = \begin{cases} \big(\Delta^M_h f\big)(x) & \text{if } x + lh \in I \text{ for } l = 0, \ldots, M, \\ 0 & \text{otherwise,} \end{cases} \tag{3.39}$$

where $x \in I$, $M \in \mathbb{N}$ and $h \in \mathbb{R}$, be the same adapted differences as in (1.287). Let v_{jm} be the hat functions according to (3.3), and let χ_{jm} be the characteristic functions of the interval I_{jm} in (3.15).

Proposition 3.5. (i) *Let $B^s_{pq}(I)$ be the same spaces as in Theorem 3.1* (i) *with $0 < p, q \le \infty$ and*

$$\frac{1}{p} < s < 1 + \min\Big(\frac{1}{p}, 1\Big), \tag{3.40}$$

Figure 3.1, p. 127. Then $f \in B^s_{pq}(I)$ can be represented as

$$f(x) = f(0)(1-x) + f(1)x - \frac{1}{2}\sum_{j=0}^{\infty}\sum_{m=0}^{2^j-1} \big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m)\, v_{jm}(x), \quad x \in I, \tag{3.41}$$

and

$$\begin{aligned} \|f \,|B^s_{pq}(I)\| &\sim \|f\,|L_p(I)\| + \Big(\int_0^1 t^{-sq} \sup_{0<h<t} \big\|\Delta^2_{h,I} f\,|L_p(I)\big\|^q \frac{\mathrm{d}t}{t}\Big)^{1/q} \\ &\sim \|f\,|L_p(I)\| + \Big(\int_0^1 h^{-sq} \big\|\Delta^2_{h,I} f\,|L_p(I)\big\|^q \frac{\mathrm{d}h}{h}\Big)^{1/q} \\ &\sim |f(0)| + |f(1)| + \Big(\int_0^1 h^{-sq} \big\|\Delta^2_{h,I} f\,|L_p(I)\big\|^q \frac{\mathrm{d}h}{h}\Big)^{1/q} \\ &\sim |f(0)| + |f(1)| + \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{1}{p})q} \Big(\sum_{m=0}^{2^j-1} |(\Delta^2_{2^{-j-1}} f)(2^{-j}m)|^p\Big)^{q/p}\Big)^{1/q} \end{aligned} \tag{3.42}$$

are equivalent quasi-norms (*with the usual modifications if $p = \infty$ and/or $q = \infty$*).

(ii) *Let $W^1_p(I)$ be the classical Sobolev spaces* (3.38) *with $1 < p < \infty$. Then $f \in W^1_p(I)$ can be represented as in* (3.41) *and*

$$\begin{aligned} \|f\,|W^1_p(I)\| &\sim \|f\,|L_p(I)\| + \|f'\,|L_p(I)\| \sim |f(0)| + |f(1)| + \|f'\,|L_p(I)\| \\ &\sim |f(0)| + |f(1)| + \Big\|\Big(\sum_{j=0}^{\infty}\sum_{m=0}^{2^j-1} 2^{2j}\big|\big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m)\,\chi_{jm}(\cdot)\big|^2\Big)^{1/2} |L_p(I)\Big\| \end{aligned} \tag{3.43}$$

are equivalent norms.

Proof. *Step* 1. It follows from Theorem 3.1 (i) that $f \in B^s_{pq}(I)$ can be represented by (3.41) and that the last line in (3.42) is an equivalent quasi-norms in $B^s_{pq}(I)$. The first and the second equivalence in (3.42) are special cases of (1.289), (1.290). As for the third equivalence it is sufficient to prove that for some $c > 0$ and all $f \in B^s_{pq}(I)$,

$$c\, \|f\,|L_p(I)\| \le |f(0)| + |f(1)| + \Big(\int_0^1 h^{-sq} \big\| \Delta^2_{h,I} f\,|L_p(I)\big\|^q \frac{\mathrm{d}h}{h} \Big)^{1/q}. \tag{3.44}$$

This is the same situation as in (3.12). The same proof by contradiction results in a function $f \in B^s_{pq}(I)$ with

$$\|f\,|L_p(I)\| = 1, \quad f(0) = f(1) = 0, \quad \Delta^2_{h,I} f(x) = 0. \tag{3.45}$$

It follows from the last assertion that f is linear in I. We refer for details in a more complicated situation to [T06, pp. 200/201]. Then f must be zero what contradicts the first assertion in (3.45).

Step 2. Part (ii) follows from Corollary 3.3 with $s = 1$ and (3.7), (3.11) with $A^1_{p,2}(I) = W^1_p(I)$. □

Remark 3.6. As a modification of part (i) of the above proposition one has for

$$B^s_{pp}(I) \quad \text{with } 1 < p < \infty,\ \tfrac{1}{p} < s < 1, \tag{3.46}$$

the equivalent norms

$$\begin{aligned} \|f\,|B^s_{pp}(I)\| &\sim \|f\,|L_p(I)\| + \Big(\int_{I\times I} \frac{|f(x)-f(y)|^p}{|x-y|^{sp+1}}\,\mathrm{d}x\,\mathrm{d}y \Big)^{1/p} \\ &\sim |f(0)| + |f(1)| + \Big(\int_{I\times I} \frac{|f(x)-f(y)|^p}{|x-y|^{sp+1}}\,\mathrm{d}x\mathrm{d}y \Big)^{1/p} \\ &\sim |f(0)| + |f(1)| + \Big(\sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} 2^{j(s-\frac{1}{p})p} \big| \big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m)\big|^p \Big)^{1/p} \end{aligned} \tag{3.47}$$

and for the Hölder spaces

$$C^s(I) = \mathcal{C}^s(I) = B^s_{\infty\infty}(I), \quad 0 < s < 1, \tag{3.48}$$

the equivalent norms

$$\begin{aligned} \|f\,|C^s(I)\| &\sim \sup_{x\in I} |f(x)| + \sup_{x,y\in I} \frac{|f(x)-f(y)|}{|x-y|^s} \\ &\sim |f(0)| + |f(1)| + \sup_{x,y\in I} \frac{|f(x)-f(y)|}{|x-y|^s} \\ &\sim |f(0)| + |f(1)| + \sup_{j,m} 2^{js} \big| \big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m)\big|. \end{aligned} \tag{3.49}$$

This follows from the above proposition and the possibility to replace $\Delta^2_{h,I} f$ for spaces with $s < 1$ by first differences $\Delta^1_{h,I} f$ according to (1.290).

Remark 3.7. In Section 3.1.1 we asked whether the Faber basis (3.1) in $C(I)$ with the expansion (3.4) is also a basis in some spaces $B^s_{pq}(I)$ and $F^s_{pq}(I)$. Theorem 3.1 (i) gives a satisfactory answer for the spaces $B^s_{pq}(I)$. For the Sobolev spaces $H^s_p(I)$ with (3.36) we have Corollary 3.3. But it remains unclear whether this assertion can be extended to spaces $H^s_p(I)$ with (3.19). The situation for the more general spaces $F^s_{pq}(I)$ is even worse. Then we have Theorem 3.1 (ii) and one may ask to which extent the restrictions according to (3.24) are natural. In any case we relied on (3.32) which reduces the question for Faber bases in $A^s_{pq}(I)$ to the corresponding question for Haar bases in $A^{s-1}_{pq}(I)$ and hence Theorem 3.1 to Theorems 2.9, 2.13. In Remarks 2.10, 2.14 we collected some references as far as Haar bases in $B^s_{pq}(I)$ are concerned. Taking these assertions for granted (independently of our approach) one can step from Haar bases to Faber bases in the way indicated above. This has been done in [Osw81] dealing with Faber bases in $B^s_{pq}(I)$, where $1 \le p < \infty$,

$$\frac{1}{p} < s < 1 + \frac{1}{p} \quad \text{or} \quad s = \frac{1}{p}, \quad 0 < q \le 1. \tag{3.50}$$

By (3.8) the restriction $q \le 1$ in the limiting case $s = 1/p$ is necessary. This limiting case is not covered by the above approach and we may ask whether J in (3.22) remains to be an isomorphic map for $B^{1/p}_{pq}(I)$, $1 \le p < \infty$, $0 < q \le 1$, onto $b^+_{pq}(I)$. Otherwise there are only a few papers in literature dealing with expansions of $f \in B^s_{pq}(I)$ in terms of Faber bases (3.41) and discrete characterisations as in the last line of (3.42) by second or higher differences. In [Cie77, Theorem 3.2, p. 401] there is a corresponding characterisation for the spaces $\mathcal{C}^s(I) = B^s_{\infty\infty}(I)$, $s > 0$. Faber expansions and related characterisations for $B^s_{pq}(I)$, $1 \le p, q \le \infty$, $\frac{1}{p} < s < 1 + \frac{1}{p}$, may be found in [CKR93, Theorem III.6, p. 189]. We could not find papers dealing with Faber bases for Sobolev spaces $H^s_p(I)$ covered by Corollary 3.3 or for spaces $F^s_{pq}(I)$ covered by Theorem 3.1 (ii). This is a little bit curious at least in case of the classical Sobolev space $W^1_p(I)$, $1 < p < \infty$, where we described the outcome in Proposition 3.5 (ii). In this special case one can argue rather directly (without the elaborated machinery on which we relied) taking the classical Littlewood–Paley characterisation for $L_p(I)$, $1 < p < \infty$, for granted, ensuring that any $g \in L_p(I)$ can be expanded by the Haar system (2.128), (2.129),

$$g(x) = \int_I g(y)\,\mathrm{d}y + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} 2^j \int_I g(y)\, h_{jm}(y)\,\mathrm{d}y\; h_{jm}(x) \tag{3.51}$$

with

$$\|g\,|L_p(I)\| \sim \left| \int_I g(y)\,\mathrm{d}y \right| + \left\| \Big(\sum_{j,m} 2^{2j} \Big| \int_I g(y)\, h_{jm}(y)\,\mathrm{d}y \Big|^2 \chi_{jm}(\cdot) \Big)^{1/2} \Big| L_p(I) \right\|. \tag{3.52}$$

If $f \in W_p^1(I)$ then it follows from (3.51) with $g = f'$ and integration that

$$f(x) = f(0)(1-x) + f(1)\,x - \frac{1}{2}\sum_{j=0}^{\infty}\sum_{m=0}^{2^j-1} \big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m)\, v_{jm}(x), \qquad (3.53)$$

where we used that

$$\int_I f'(y)\, h_{jm}(y)\, \mathrm{d}y = -\big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m), \quad j \in \mathbb{N}_0;\ m = 0,\ldots,2^j-1, \ (3.54)$$

and

$$\int_0^x h_{jm}(y)\, \mathrm{d}y = 2^{-j-1}\, v_{jm}(x), \quad j \in \mathbb{N}_0,\ m = 0,\ldots,2^j-1. \qquad (3.55)$$

Furthermore, (3.52) with $g = f'$ and (3.54) give (3.43). In other words, the representation (3.53) = (3.41) and the equivalent norms for functions $f \in W_p^1(I)$ can be obtained by rather straightforward calculations from the orthogonal Haar basis (2.128) and the Littlewood–Paley theorem for $L_p(I)$ with $1 < p < \infty$. *This should have been known for ages, but we have no references.*

3.2 Faber bases on cubes

3.2.1 Preliminaries and definitions

We dealt in Section 2.4 with Haar tensor systems on cubes $\mathbb{Q}^n$ according to (2.313) with a preference of $n = 2$, hence the square

$$\mathbb{Q}^2 = \big\{x = (x_1,x_2) \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < 1\big\}. \qquad (3.56)$$

For this purpose we introduced the Haar tensor system (2.284)–(2.286) on $\mathbb{Q}^2$ as the product of the Haar system (2.282), (2.283) on the interval $I = (0,1)$. Now we wish to do the same with the Faber system (3.1)–(3.3) on I, hence

$$\big\{v_0, v_1, v_{jm} : j \in \mathbb{N}_0;\ m = 0,\ldots,2^j-1\big\} \qquad (3.57)$$

with the basic functions

$$v_0(x) = 1 - x, \quad v_1(x) = x \quad \text{where } 0 \le x \le 1, \qquad (3.58)$$

and the hat-functions $v_{jm}(x)$, $0 \le x \le 1$,

$$v_{jm}(x) = \begin{cases} 2^{j+1}\big(x - 2^{-j}m\big) & \text{if } 2^{-j}m \le x < 2^{-j}m + 2^{-j-1}, \\ 2^{j+1}\big(2^{-j}(m+1) - x\big) & \text{if } 2^{-j}m + 2^{-j-1} \le x < 2^{-j}(m+1), \\ 0 & \text{otherwise.} \end{cases} \qquad (3.59)$$

This coincides also with (2.3)–(2.5), Figure 2.1, p. 64, repeated here for sake of convenience. Let $x = (x_1, x_2) \in \mathbb{Q}^2$ and

$$v_{km}(x) = \begin{cases} v_{m_1}(x_1)\, v_{m_2}(x_2) \\ \qquad \text{if } k = (-1,-1);\ m_1 \in \{0,1\},\ m_2 \in \{0,1\}, \\ v_{m_1}(x_1)\, v_{k_2 m_2}(x_2) \\ \qquad \text{if } k = (-1,k_2),\ k_2 \in \mathbb{N}_0;\ m_1 \in \{0,1\},\ m_2 = 0,\ldots,2^{k_2}-1, \\ v_{k_1 m_1}(x_1)\, v_{m_2}(x_2) \\ \qquad \text{if } k = (k_1,-1),\ k_1 \in \mathbb{N}_0;\ m_1 = 0,\ldots,2^{k_1}-1,\ m_2 \in \{0,1\}, \\ v_{k_1 m_1}(x_1)\, v_{k_2 m_2}(x_2) \\ \qquad \text{if } k \in \mathbb{N}_0^2;\ m_l = 0,\ldots,2^{k_l}-1;\ l = 1,2. \end{cases} \tag{3.60}$$

Recall that $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\} = \{-1,0,1,\ldots\}$ and $\mathbb{N}^2_{-1} = \mathbb{N}_{-1} \times \mathbb{N}_{-1}$. Let

$$\mathbb{P}^F_k = \big\{ m \in \mathbb{Z}^2 \text{ with } m \text{ as in (3.60)} \big\}, \quad k \in \mathbb{N}^2_{-1}. \tag{3.61}$$

Then

$$\big\{ v_{km} : k \in \mathbb{N}^2_{-1},\ m \in \mathbb{P}^F_k \big\} \tag{3.62}$$

is the *Faber system* on $\mathbb{Q}^2$ we are looking for. It is the tensor product of (3.58), (3.59) and also the counterpart of the Haar tensor system (2.284)–(2.286). As explained in Section 3.1.1 to call (3.62) a Faber tensor system is not necessary because there is nothing what should be denoted as Faber wavelet system, in contrast to related Haar systems. Faber systems apply to spaces $S^r_{pq}B(\mathbb{Q}^2)$ and $S^r_p H(\mathbb{Q}^2)$ having boundary values at $\partial\mathbb{Q}^2$. This complicates the situation somewhat (compared with Haar systems). For this reason we split our considerations dealing first with spaces $S^r_{pp}B(\mathbb{Q}^2)$, $S^1_p W(\mathbb{Q}^2)$ where we have tensor arguments and later on with the above more general spaces. First we introduce sequence spaces covering all cases. For this purpose we modify Definition 2.40 as follows. Let χ_{km} be as in Definition 2.30 the characteristic function of the rectangle

$$Q_{km} = \big(2^{-k_1}m_1,\ 2^{-k_1}(m_1+1)\big) \times \big(2^{-k_2}m_2,\ 2^{-k_2}(m_2+1)\big), \tag{3.63}$$

and

$$\chi^{(p)}_{km}(x) = 2^{\frac{k_1+k_2}{p}}\, \chi_{km}(x), \quad k \in \mathbb{N}^2_0,\ m \in \mathbb{P}^F_k, \tag{3.64}$$

its p-normalisation, $0 < p \le \infty$. Recall that $I = (0,1)$ is the unit interval on $\mathbb{R}$.

Definition 3.8. Let $0 < p,q \le \infty$. Then $s^F_{pq}b(\mathbb{Q}^2)$ is the collection of all complex sequences

$$\lambda = \big\{ \lambda_{km} : k \in \mathbb{N}^2_{-1},\ m \in \mathbb{P}^F_k \big\} \tag{3.65}$$

with

$$\|\lambda \,|s^F_{pq}b(\mathbb{Q}^2)\| = \Big(\sum_{k \in \mathbb{N}^2_{-1}} \Big(\sum_{m \in \mathbb{P}^F_k} |\lambda_{km}|^p \Big)^{q/p} \Big)^{1/q} < \infty \tag{3.66}$$

and $s^F_{pq} f(\mathbb{Q}^2)$ is the collection of all complex sequences

$$\begin{aligned} \lambda = &\big\{\lambda_{km} : k \in \mathbb{N}_0^2,\ m \in \mathbb{P}_k^F\big\} \\ &\cup \big\{\lambda^r_{jl} : j \in \mathbb{N}_0,\ l = 0, \ldots, 2^j - 1;\ r = 1,2,3,4\big\} \\ &\cup \big\{\lambda^{r_1,r_2} : r_1 \in \{0,1\},\ r_2 \in \{0,1\}\big\} \end{aligned} \tag{3.67}$$

with

$$\begin{aligned} \|\lambda \,|s^F_{pq} f(\mathbb{Q}^2)\| = &\Big\| \Big(\sum_{k \in \mathbb{N}_0^2} \sum_{m \in \mathbb{P}_k^F} |\lambda_{km}\, \chi^{(p)}_{km}(\cdot)|^q \Big)^{1/q} |L_p(\mathbb{Q}^2)\Big\| \\ &+ \sum_{r=1}^{4} \Big\| \Big(\sum_{j \in \mathbb{N}_0} \sum_{l=0}^{2^j-1} |\lambda^r_{jl}\, \chi^{(p)}_{jl}(\cdot)|^q \Big)^{1/q} |L_p(I)\Big\| \\ &+ |\lambda^{0,0}| + |\lambda^{1,0}| + |\lambda^{0,1}| + |\lambda^{1,1}| < \infty \end{aligned} \tag{3.68}$$

with the usual modifications if $p = \infty$ and/or $q = \infty$.

Remark 3.9. The second and third terms in (3.68) refer to the sides and to the corner points of $\mathbb{Q}^2$ and originate from $f^+_{pq}(I)$ in (3.18) where $\chi^{(p)}_{jl}$ has the same meaning as there. With obvious identifications one has

$$s^F_{pp} b(\mathbb{Q}^2) = s^F_{pp} f(\mathbb{Q}^2), \quad 0 < p \le \infty, \tag{3.69}$$

and

$$s^F_{p,\min(p,q)} b(\mathbb{Q}^2) \hookrightarrow s^F_{pq} f(\mathbb{Q}^2) \hookrightarrow s^F_{p,\max(p,q)} b(\mathbb{Q}^2). \tag{3.70}$$

3.2.2 Faber bases in $C(\mathbb{Q}^2)$

Recall that the Faber system (3.1)–(3.3) is a basis in $C(I)$ with the expansion (3.4), (3.5). This assertion can be extended from $C(I)$ to $C(\mathbb{Q}^n)$, where $\mathbb{Q}^n$ is the unit cube in $\mathbb{R}^n$ according to (2.313). It is sufficient to deal with $n = 2$, hence the space $C(\mathbb{Q}^2)$ where $\mathbb{Q}^2$ is the square (3.56). The extension to $C(\mathbb{Q}^n)$ is a technical matter and will not be done here in detail. According to Definition 1.24 (iii) the Banach space $C(\mathbb{Q}^2)$ is the collection of all complex-valued continuous functions in $\overline{\mathbb{Q}^2} = [0,1] \times [0,1]$, furnished in the usual way with the norm

$$\|f \,|C(\mathbb{Q}^2\| = \sup_{x \in \mathbb{Q}^2} |f(x)| = \max_{x \in \overline{\mathbb{Q}^2}} |f(x)|. \tag{3.71}$$

We need a few notation. We specify (1.163), (1.164) by

$$\Delta^2_{h,1} f(x_1, x_2) = f(x_1 + 2h, x_2) - 2f(x_1 + h, x_2) + f(x_1, x_2), \tag{3.72}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $h \in \mathbb{R}$, similarly for $\Delta^2_{h,2} f(x_1, x_2)$, and the second mixed differences

$$\Delta^{2,2}_{h_1,h_2} f(x_1, x_2) = \Delta^2_{h_2,2}\big(\Delta^2_{h_1,1} f\big)(x_1, x_2), \quad h_1 \in \mathbb{R},\ h_2 \in \mathbb{R}, \tag{3.73}$$

evaluating the continuous function f in $\mathbb{R}^2$ at the points $(x_1 + l_1 h_1, x_2 + l_2 h_2)$ with $l_1, l_2 \in \{0, 1, 2\}$. We extend now (3.5) from I to $\mathbb{Q}^2$. Let $k \in \mathbb{N}^2_{-1}$ and $m \in \mathbb{P}^F_k$ according to (3.61) and $f \in C(\mathbb{Q}^2)$. Then

$$d^2_{km}(f) = f(m_1, m_2) \quad \text{if } k = (-1, -1),\ m_1 \in \{0, 1\},\ m_2 \in \{0, 1\}, \tag{3.74}$$

$$\begin{aligned} d^2_{km}(f) = &-\tfrac{1}{2}\Delta^2_{2^{-k_2-1},2} f\big(m_1, 2^{-k_2} m_2\big) \\ &\text{if } k = (-1, k_2),\ k_2 \in \mathbb{N}_0,\ m_1 \in \{0, 1\},\ m_2 = 0, \ldots, 2^{k_2} - 1, \end{aligned} \tag{3.75}$$

$$\begin{aligned} d^2_{km}(f) = &-\tfrac{1}{2}\Delta^2_{2^{-k_1-1},1} f\big(2^{-k_1} m_1, m_2\big) \\ &\text{if } k = (k_1, -1),\ k_1 \in \mathbb{N}_0,\ m_2 \in \{0, 1\},\ m_1 = 0, \ldots, 2^{k_1} - 1, \end{aligned} \tag{3.76}$$

$$\begin{aligned} d^2_{km}(f) = &\tfrac{1}{4}\Delta^{2,2}_{2^{-k_1-1},2^{-k_2-1}} f\big(2^{-k_1} m_1, 2^{-k_2} m_2\big) \\ &\text{if } k \in \mathbb{N}^2_0,\ m_l = 0, \ldots, 2^{k_l} - 1;\ l = 1, 2. \end{aligned} \tag{3.77}$$

Theorem 3.10. *The Faber system (3.62) is a basis in $C(\mathbb{Q}^2)$,*

$$f(x) = \sum_{K=0}^{\infty} \big(f_{K+1}(x) - f_K(x)\big) + f_0(x), \quad f \in C(\mathbb{Q}^2), \tag{3.78}$$

convergence in $C(\mathbb{Q}^2)$, with

$$f_K(x) = \sum_{\substack{k \in \mathbb{N}^2_{-1} \\ k_1 \le K,\, k_2 \le K}} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(f)\, v_{km}(x), \quad K \in \mathbb{N}_0. \tag{3.79}$$

Proof. For fixed $x_2 \in [0, 1]$ we expand $x_1 \mapsto f(x_1, x_2)$ according to (3.4),

$$\begin{aligned} f(x_1, x_2) = &f(0, x_2)\, v_0(x_1) + f(1, x_2)\, v_1(x_1) \\ &- \frac{1}{2} \sum_{j=0}^{\infty} \sum_{l=0}^{2^j - 1} \big(\Delta^2_{2^{-j-1},1} f\big)(2^{-j} l, x_2)\, v_{jl}(x_1). \end{aligned} \tag{3.80}$$

Expanding the coefficients

$$f(0, x_2), \quad f(1, x_2), \quad -\tfrac{1}{2}\big(\Delta^2_{2^{-j-1},1} f\big)(2^{-j} l, x_2) \tag{3.81}$$

according to (3.4) with respect to x_2 one obtains

$$f(x) = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(f)\, v_{km}(x), \quad x \in \mathbb{Q}^2, \tag{3.82}$$

at least formally. The interpretation of (3.82) by (3.78), (3.79) can be justified as follows. For $J \in \mathbb{N}_0$ and fixed $x_2 \in [0,1]$ let

$$\begin{aligned} f^J(x_1,x_2) = {}& f(0,x_2)\,v_0(x_1) + f(1,x_2)\,v_1(x_1) \\ & - \frac{1}{2}\sum_{j=0}^{J}\sum_{l=0}^{2^j-1} \big(\Delta^2_{2^{-j-1},1} f\big)(2^{-j}l, x_2)\, v_{jl}(x_1) \end{aligned} \tag{3.83}$$

be the corresponding partial sums of (3.4). By (2.26), (2.27) we have

$$f^J(x_1,x_2) = f(x_1,x_2) \quad \text{if } x_1 = 2^{-J-1}m \ \text{ with } m = 0,\ldots,2^{J+1}, \tag{3.84}$$

for $0 \le x_2 \le 1$. Expanding the functions in (3.81) where $j = 0,\ldots,J$, with respect to x_2 one obtains the counterparts of (3.84) and, as a consequence,

$$f_K(x) = f(x) \quad \text{if } x = \big(2^{-K-1}m_1,\, 2^{-K-1}m_2\big),\ 0 \le m_1, m_2 \le 2^{K+1}. \tag{3.85}$$

In the corresponding squares with side-length 2^{-K-1} the function $f_K(x)$ is linear in x_1 for fixed x_2 and vice versa. Then (3.85) and the uniform continuity of f prove (3.78). It remains to prove that the representation (3.78), (3.79) is unique. Let

$$0 = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^F_k} a_{km}\, v_{km}(x), \quad x \in \overline{\mathbb{Q}^2}, \tag{3.86}$$

convergence in $C(\mathbb{Q}^2)$ as in (3.78), (3.79). For $x_2 = 0$ one has the one-dimensional situation considered in (2.34) and the corresponding coefficients are zero. The same for $x_2 = 1$. Afterwards one chooses $x_2 = 1/2$ with the same outcome. Then one obtains $a_{km} = 0$ by iteration. Hence, $\{v_{km}\}$ is a basis. □

Remark 3.11. The above proof applies to K-blocks in (3.78), (3.79). But in which way one orders the basis $\{v_{km}\}$ within these blocks does not matter. This is not totally obvious but it follows from the above arguments. The above theorem is more or less a routine extension of Faber's assertion according to Theorem 2.1 (iii) proved 100 years ago in [Fab09] and generalised by Schauder in [Scha27] (one-dimensional case). We refer also to our historical comments in Remark 2.3. We did not find the above theorem explicitly stated in literature. But it follows from more general assertions about product bases in spaces of type $C(\mathbb{Q}^2)$ in [Sem63], [Sem82].

Remark 3.12. We explained between Definition 1.17 and Theorem 1.18 what is meant by a conditional basis in a quasi-Banach space. As formulated in Theorem 2.1 (iii) the Faber system (3.1) is a conditional basis in $C(I)$ because any basis in $C(I)$ is conditional, [Woj91, Corollary 12, p. 63]. Furthermore, it follows from the famous theorem by Milutin, which may be found in [Woj91, Theorem, p. 160], that $C(\mathbb{Q}^2)$ is isomorphic to $C(I)$. Hence, any basis in $C(\mathbb{Q}^2)$ is conditional. This applies in particular to the Faber basis (3.62).

3.2.3 Faber bases in $S^1_p W(\mathbb{Q}^2)$ and $S^r_{pp}B(\mathbb{Q}^2)$

Let $\mathbb{Q}^2$ be the square according to (3.56) and let $S^r_{pq}A(\mathbb{Q}^2)$ be the spaces with dominating mixed smoothness as introduced in Definition 1.56 by restriction of $S^r_{pq}A(\mathbb{R}^2)$ to $\mathbb{Q}^2$. Here $A \in \{B, F\}$, $0 < p, q \le \infty$ ($p < \infty$ for F-spaces), and $r \in \mathbb{R}$. Recall

$$S^r_{pq}A(\mathbb{Q}^2) \hookrightarrow C(\mathbb{Q}^2) \quad \text{if } 0 < p, q \le \infty,\ r > 1/p, \tag{3.87}$$

(again with $p < \infty$ for F-spaces). This follows by restriction to $\mathbb{Q}^2$ of a corresponding assertion for $\mathbb{R}^2$ which is covered by [ST87, pp. 132/133]. One may compare (3.87) with the non-limiting cases in (3.8), (3.9). For Haar tensor bases we have the Theorems 2.32, 2.38 and in particular 2.41 which apply to the same $\left(\frac{1}{p}, r\right)$-regions as for Haar bases on intervals, Figure 2.3, p. 82 (with $r = s$). Now we wish to do the same with respect to the Faber system (3.62) for the spaces $S^r_{pq}B(\mathbb{Q}^2)$ and $S^r_p H(\mathbb{Q}^2)$, where p and r are restricted as in case of $B^s_{pq}(I)$ and $H^s_p(I)$, Theorem 3.1, Corollary 3.3, Figure 3.1, p. 127 with p and $s = r$. Similarly as in case of Haar tensor bases we split the considerations dealing first with the Besov spaces $S^r_{pp}B(\mathbb{Q}^2)$ and the Sobolev spaces $S^1_p W(\mathbb{Q}^2)$ with dominating mixed smoothness. For these spaces we have some additional information. We refer to Section 1.2.8, in particular to Theorems 1.67, 1.70 and Proposition 1.68. Furthermore we constructed in Theorem 1.66 wavelet bases in $S^r_{pp}B(\mathbb{Q}^2)$ where we relied on Theorem 1.58. Now we are doing the same with respect to the Faber bases $\{v_0, v_1, v_{jm}\}$ in (3.1) on $I = (0,1)$ and $\{v_{km}\}$ in (3.62) on $\mathbb{Q}^2$.

Theorem 3.13. *Let $\mathbb{Q}^2$ be the square (3.56) and let $\{v_{km}\}$ be the Faber system (3.60)–(3.62). Let $d^2_{km}(f)$ be the mixed differences according to (3.74)–(3.77). Let $s^F_{pq}b(\mathbb{Q}^2)$ and $s^F_{pq}f(\mathbb{Q}^2)$ be the sequence spaces as introduced in Definition 3.8.*

(i) *Let $S^r_{pp}B(\mathbb{Q}^2)$ with*

$$0 < p \le \infty, \quad \frac{1}{p} < r < 1 + \min\left(\frac{1}{p}, 1\right), \tag{3.88}$$

be the above spaces, Figure 3.1, p. 127 (with r in place of s). Let $f \in D'(\mathbb{Q}^2)$ (or likewise $f \in L_1(\mathbb{Q}^2)$). Then $f \in S^r_{pp}B(\mathbb{Q}^2)$ if, and only if, it can be represented as

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^F_k} \lambda_{km} 2^{-(k_1+k_2)(r-\frac{1}{p})}\, v_{km}, \quad \lambda \in s^F_{pp}b(\mathbb{Q}^2), \tag{3.89}$$

unconditional convergence being in $L_1(\mathbb{Q}^2)$. The representation (3.89) is unique with

$$\lambda_{km} = \lambda_{km}(f) = 2^{(k_1+k_2)(r-\frac{1}{p})}\, d^2_{km}(f), \quad k \in \mathbb{N}^2_{-1},\ m \in \mathbb{P}^F_k, \tag{3.90}$$

and

$$J : f \mapsto \lambda(f) \tag{3.91}$$

is an isomorphic map of $S^r_{pp}B(\mathbb{Q}^2)$ onto $s^F_{pp}b(\mathbb{Q}^2)$. If $p < \infty$ then $\{v_{km}\}$ is an unconditional basis in $S^r_{pp}B(\mathbb{Q}^2)$.

(ii) *Let* $1 < p < \infty$. *Let* $f \in D'(\mathbb{Q}^2)$ (*or likewise* $f \in L_1(\mathbb{Q}^2)$). *Then* $f \in S^1_p W(\mathbb{Q}^2)$ *if, and only if, it can be represented as*

$$f = \sum_{k\in\mathbb{N}^2_{-1}} \sum_{m\in\mathbb{P}^F_k} \lambda_{km}\, 2^{-(k_1+k_2)(1-\frac{1}{p})}\, v_{km}, \quad \lambda \in s^F_{p,2} f(\mathbb{Q}^2), \tag{3.92}$$

in the interpretation given in Remark 3.9, unconditional convergence being in $L_1(\mathbb{Q}^2)$. *The representation (3.92) is unique with (3.90) and* J *in (3.91) is an isomorphic map of* $S^1_p W(\mathbb{Q}^2)$ *onto* $s^F_{p,2} f(\mathbb{Q}^2)$. *Furthermore,* $\{v_{km}\}$ *is an unconditional basis in* $S^1_p W(\mathbb{Q}^2)$ *and*

$$S^1_{p,\min(p,2)} B(\mathbb{Q}^2) \hookrightarrow S^1_p W(\mathbb{Q}^2) \hookrightarrow S^1_{p,\max(p,2)} B(\mathbb{Q}^2) \hookrightarrow C(\mathbb{Q}^2). \tag{3.93}$$

Proof. Step 1. We prove part (i). We are in the same position as in the proof of the parts (i) and (ii) of Theorem 1.66. Let $f \in S^r_{pp} B(\mathbb{R}^2)$ now with r as in (3.88). Then we have (1.279), (1.280) with $I^2 = \mathbb{R} \times I$ as there where $I = (0,1)$ is again the unit interval on $\mathbb{R}$. Now we expand $f_{k_1 m_1} \in B^r_{pp}(I)$ according to Theorem 3.1 (i) with $r = s$ and

$$f_{k_1 m_1}(x_2) = \lambda^{k_1 m_1}_0\, v_0(x_2) + \lambda^{k_1 m_1}_1\, v_1(x_2) + \sum_{k_2=0}^{\infty} \sum_{m_2=0}^{2^{k_2}-1} \lambda^{k_1 k_2}_{m_1 m_2}\, 2^{-k_2(r-\frac{1}{p})}\, v_{k_2 m_2}(x_2) \tag{3.94}$$

as the adapted substitute of (1.281). But otherwise we can argue as there and obtain related counterparts of (1.282)–(1.286). This gives the representation (3.89). Using the properties of v_{km} it follows from (3.89) that

$$\begin{aligned} |f(x)| &\le \sum_{k\in\mathbb{N}^2_{-1}} 2^{-(k_1+k_2)(r-\frac{1}{p})} \sup_{m\in\mathbb{P}^F_k} |\lambda_{km}| \\ &\le \|\lambda | s^F_{pp}(\mathbb{Q}^2)\| \sum_{k\in\mathbb{N}^2_{-1}} 2^{-(k_1+k_2)(r-\frac{1}{p})} < \infty \end{aligned} \tag{3.95}$$

where we used $r > 1/p$. Together with $v_{km} \in C(\mathbb{Q}^2)$ one obtains that (3.89) converges also in $C(\mathbb{Q}^2)$. This proves again (3.87) but also (3.90) as a consequence of Theorem 3.10. Finally one can argue as at the end of Step 1 of the proof of Theorem 1.66 that J in (3.91) is not only a map into $s^F_{pp} b(\mathbb{Q}^2)$, but a map onto $s^F_{pp} b(\mathbb{Q}^2)$. If $p < \infty$ then $\{v_{km}\}$ is an unconditional basis in $S^r_{pp} B(\mathbb{Q}^2)$.

Step 2. We prove part (ii). So far we have the equivalent norm (1.309) for the Sobolev spaces $S^1_p W(\mathbb{Q}^2)$ with dominating mixed smoothness and the properties listed in Proposition 1.68. The last embedding in (3.93) is covered by (3.87). Otherwise, (3.93) follows from the restriction of

$$S^1_{p,\min(p,2)} B(\mathbb{R}^2) \hookrightarrow S^1_p W(\mathbb{R}^2) = S^1_{p,2} F(\mathbb{R}^2) \hookrightarrow S^1_{p,\max(p,2)} B(\mathbb{R}^2) \tag{3.96}$$

to $\mathbb{Q}^2$. As far as (3.96) is concerned we refer to [ST87, pp. 88/89]. It follows also from the wavelet representation in Theorem 1.54 and (3.70) with $\mathbb{R}^2$ in place of $\mathbb{Q}^2$. Let f be given by (3.92). Using (3.70) it follows in the same way as in (3.95) that (3.92) converges unconditionally in $C(\mathbb{Q}^2)$ and hence also in $L_1(\mathbb{Q}^2)$. In particular the representation (3.92) is unique with λ_{km} as in (3.90) and $f \in C(\mathbb{Q}^2)$. We simplify temporarily $\mathbb{Q}^2$ by $Q = \mathbb{Q}^2$. Based on (3.60) we split f in (3.92) into an interior part

$$f^Q = \sum_{k \in \mathbb{N}_0^2} \sum_{m \in \mathbb{P}_k^F} \lambda_{km} \, 2^{-(k_1+k_2)(1-\frac{1}{p})} v_{km} \tag{3.97}$$

(having boundary values zero at ∂Q), a boundary part

$$f^{\partial Q} = v_0(x_1) \sum_{k_2=0}^{\infty} \sum_{m_2=0}^{2^{k_2}-1} \lambda^1_{k_2 m_2} \, 2^{-(k_2-1)(1-\frac{1}{p})} \, v_{k_2 m_2}(x_2) + ++ \tag{3.98}$$

(zero in the corner points) and a corner part

$$f^{\partial^2 Q} = 2^{2(1-\frac{1}{p})} \sum_{m_1=0}^{1} \sum_{m_2=0}^{1} \lambda^{m_1,m_2} \, v_{m_1}(x_1) \, v_{m_2}(x_2), \tag{3.99}$$

where we adapted the notation to (3.67), (3.68) and the comments in Remark 3.9, such that

$$f = f^Q + f^{\partial Q} + f^{\partial^2 Q}. \tag{3.100}$$

Here $+++$ in (3.98) refers to three further terms with $v_1(x_1)$, $v_0(x_2)$, $v_1(x_2)$ in place of $v_0(x_1)$ according to (3.60). Obviously, $f^{\partial^2 Q} \in S^1_p W(Q)$ and

$$\begin{aligned} \sum_{m_1=0}^{1} \sum_{m_2=0}^{1} |\lambda^{m_1,m_2}| &\sim \| f^{\partial^2 Q} | C(Q) \| \\ &\le c \, \| f^{\partial^2 Q} | S^1_p W(Q) \| \\ &\le c' \sum_{m_1=0}^{1} \sum_{m_2=0}^{1} |\lambda^{m_1,m_2}|. \end{aligned} \tag{3.101}$$

By Theorem 3.1 (ii) the x_2-factor of the first term in (3.98) belongs to $W^1_p(I) = H^1_p(I) = F^1_{p,2}(I)$, Figure 3.1, p. 127, with a corresponding equivalent norm. Multiplication with $v_0(x_1)$, corresponding assertions for the three other terms in (3.98), and the equivalent norms (1.309), (1.312) prove $f^{\partial Q} \in S^1_p W(Q)$ and

$$\| f^{\partial Q} | S^1_p W(Q) \| \sim \sum_{r=1}^{4} \Big\| \Big(\sum_{j=0}^{\infty} \sum_{l=0}^{2^j-1} |\lambda^r_{jl} \, \chi^{(p)}_{jl}(\cdot)|^2 \Big)^{1/2} | L_p(I) \Big\|, \tag{3.102}$$

adapted to (3.68). By (3.60) and the Haar tensor system $\{h_{km}\}$ according to (2.284) we have

$$\frac{\partial^2 v_{km}(x)}{\partial x_1 \partial x_2} = 2^{k_1+k_2+2}\, h_{km}(x), \quad k \in \mathbb{N}_0^2,\ m \in \mathbb{P}_k^F = \mathbb{P}_k^H,\ x \in Q. \tag{3.103}$$

Then it follows from (3.97) and Theorem 2.41 (ii) applied to $L_p(Q) = S_p^0 H(Q)$ that

$$\frac{\partial^2 f^Q}{\partial x_1 \partial x_2} = 4 \sum_{k \in \mathbb{N}_0^2} \sum_{m \in \mathbb{P}_k^F} \lambda_{km}\, 2^{\frac{k_1+k_2}{p}} \in L_p(Q) \tag{3.104}$$

and

$$\Big\| \frac{\partial^2 f^Q}{\partial x_1 \partial x_2} | L_p(Q) \Big\| \sim \Big\| \Big(\sum_{k \in \mathbb{N}_0^2} \sum_{m \in \mathbb{P}_k^F} |\lambda_{km}\, \chi_{km}^{(p)}(\cdot)|^2 \Big)^{1/2} | L_p(Q) \Big\| < \infty, \tag{3.105}$$

where we used $\lambda \in s_{p,2}^F f(Q)$. Since $f^Q | \partial Q = 0$ one has by (1.312) that $f^Q \in S_p^1 W(Q)$ and

$$\| f^Q \,| S_p^1 W(Q) \| \sim \Big\| \frac{\partial^2 f^Q}{\partial x_1 \partial x_2} \,| L_p(Q) \Big\|. \tag{3.106}$$

More precisely: First one has this equivalence for finite sums of f^Q in (3.97) belonging to $S_p^1 W(Q)$ and obtains afterwards $f^Q \in S_p^1 W(Q)$ and (3.106) by completion. Then one has by (3.101), (3.102) and (3.106) that $f \in S_p^1 W(Q)$ and

$$\| f \,| S_p^1 W(Q) \| \le c\, \| \lambda \,| s_{p,2}^F f(Q) \|. \tag{3.107}$$

We prove the converse. By (3.93) and Theorem 3.10 one can represent $f \in S_p^1 W(Q)$ as

$$\begin{aligned} f = \sum_{k \in \mathbb{N}_{-1}^2} \sum_{m \in \mathbb{P}_k^F} d_{km}^2(f)\, v_{km} &= \sum_{k \in \mathbb{N}_{-1}^2} \sum_{m \in \mathbb{P}_k^F} \lambda_{km}\, 2^{-(k_1+k_2)(1-\frac{1}{p})}\, v_{km} \\ &= f^Q + f^{\partial Q} + f^{\partial^2 Q}, \end{aligned} \tag{3.108}$$

decomposed as before in (3.97)–(3.99). By (3.93) one has

$$\sum_{m_1=0}^{1} \sum_{m_2=0}^{1} |\lambda^{m_1,m_2}| \le c\, \| f \,| C(Q) \| \le c'\, \| f \,| S_p^1 W(Q) \|. \tag{3.109}$$

From (1.311) and Theorem 3.1 (ii) with $W_p^1(I) = F_{p,2}^1(I)$, Figure 3.1, p. 127, it follows that

$$\sum_{r=1}^{4} \Big\| \Big(\sum_{j=0}^{\infty} \sum_{l=0}^{2^j-1} |\lambda_{jl}^r\, \chi_{jl}^{(p)}(\cdot)|^2 \Big)^{1/2} | L_p(I) \Big\| \le c\, \| f \,| W_p^1(\partial Q) \| \le c'\, \| f \,| S_p^1 W(Q) \|. \tag{3.110}$$

By the above considerations one has again $f^{\partial^2 Q} \in S^1_p W(Q)$ and $f^{\partial Q} \in S^1_p W(Q)$. Then $f^Q = f - f^{\partial^2 Q} - f^{\partial Q} \in S^1_p W(Q)$ and by (1.312)

$$\| f^Q \,|S^1_p W(Q)\| \sim \left\| \frac{\partial^2 f^Q}{\partial x_1 \partial x_2} \,|L_p(Q)\right\| \tag{3.111}$$

with the equivalence (3.105). Hence $f \in S^1_p W(Q)$ can be represented by (3.92) and

$$\|\lambda \,|s^F_{p,2} f(Q)\| \le c \,\| f \,|S^1_p W(Q)\|. \tag{3.112}$$

Together with (3.107) it follows that J in (3.91) is an isomorphic map of $S^1_p W(Q)$ onto $s^F_{p,2} f(Q)$. □

For later purposes we collect, complement and specify some previous assertions. This might be considered as the counterpart of Proposition 3.5. Let as before $\mathbb{Q}^2$ be the square according to (3.56), sometimes shortened as $Q = \mathbb{Q}^2$. Recall that $I = (0,1)$ is the unit interval on the real line $\mathbb{R}$. Let

$$\Delta^2_{h,I} f, \quad \Delta^2_{h,1,Q} f, \quad \Delta^2_{h,2,Q} f, \quad h \in \mathbb{R}, \tag{3.113}$$

and

$$\Delta^{2,2}_{h,Q} f, \quad h = (h_1, h_2) \in \mathbb{R}^2, \tag{3.114}$$

be the same adapted differences as in (1.287), (1.291) and (1.293) with $M = 2$. Let $S^r_{pp} B(\mathbb{Q}^2)$ and $S^1_p W(\mathbb{Q}^2)$ be the same spaces as in the above theorem. Let χ_{km} be the characteristic function of the rectangle Q_{km} in (3.63) and χ_{jl} be the characteristic function of the interval I_{jl} in (3.15). Let $W^1_p(\partial Q)$ and $\mathrm{tr}_{\partial Q}$ be as in (1.306) and (1.307).

Corollary 3.14. *Let $\mathbb{Q}^2 = Q$ be the square (3.56) in $\mathbb{R}^2$. Let $\{v_{km}\}$ be the Faber system (3.60)–(3.62) and let $d^2_{km}(f)$ be the mixed differences according to (3.74)–(3.77).*

(i) *Let*

$$0 < p \le \infty, \quad \frac{1}{p} < r < 1 + \min\left(\frac{1}{p}, 1\right). \tag{3.115}$$

Then $f \in S^r_{pp} B(Q)$ can be represented as

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(f)\, v_{km} \tag{3.116}$$

with

$$\begin{aligned}
\| f \,|S^r_{pp} B(Q)\| &\sim \Big(\sum_{k \in \mathbb{N}^2_{-1}} 2^{(k_1+k_2)(r-\frac{1}{p})p} \sum_{m \in \mathbb{P}^F_k} |d^2_{km}(f)|^p \Big)^{1/p} \\
&\sim \| f \,|L_p(Q)\| + \Big(\int_0^1 h^{-rp} \Big(\big\| \Delta^2_{h,1,Q} f \,|L_p(Q)\big\|^p \\
&\qquad + \big\| \Delta^2_{h,2,Q} f \,|L_p(Q)\big\|^p \Big) \frac{\mathrm{d}h}{h} \Big)^{1/p} \\
&\quad + \Big(\int_0^1 \int_0^1 (h_1 h_2)^{-rp} \big\| \Delta^{2,2}_{h,Q} f \,|L_p(Q)\big\|^p \,\frac{\mathrm{d}h}{h_1 h_2} \Big)^{1/p},
\end{aligned} \tag{3.117}$$

(*equivalent quasi-norms with the usual modification if* $p = \infty$).

(ii) *Let* $1 < p < \infty$. *Then* $f \in S^1_p W(Q)$ *can be represented by* (3.116) *with*

$$
\begin{aligned}
\|f \,|S^1_p W(Q)\| \sim & \sum_{\substack{m_1 \in \{0,1\} \\ m_2 \in \{0,1\}}} |f(m_1, m_2)| \\
& + \Big\| \Big(\sum_{j=0}^{\infty} \sum_{l=0}^{2^j - 1} 2^{2j} \big| \Delta^2_{2^{-j-1},2} f(0, 2^{-j} l)\, \chi_{jl}(\cdot) \big|^2 \Big)^{1/2} |L_p(I) \Big\| + {+}{+} \\
& + \Big\| \Big(\sum_{k \in \mathbb{N}^2_0} \sum_{m \in \mathbb{P}^F_k} 2^{2(k_1+k_2)} |d^2_{km}(f)\, \chi_{km}(\cdot)|^2 \Big)^{1/2} |L_p(Q) \Big\| \\
\sim & \, \|f \,|L_p(Q)\| + \sum_{l=1}^{2} \Big\| \frac{\partial f}{\partial x_l} \,|L_p(Q) \Big\| + \Big\| \frac{\partial^2 f}{\partial x_1 \partial x_2} \,|L_p(Q) \Big\| \\
\sim & \Big(\int_Q \Big| \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} \Big|^p \mathrm{d}x \Big)^{1/p} + \big\| \mathrm{tr}_{\partial Q} f \,|W^1_p(\partial Q) \big\|
\end{aligned}
\tag{3.118}
$$

(*equivalent norms*).

Proof. The representation (3.116) is a reformulation of (3.89), (3.90). The first equivalences in (3.117), (3.118) follow from the isomorphic maps in Theorem 3.13 and the related sequence spaces according to Definition 3.8. Here the second term on the right-hand side of (3.118) refers to the side $(0, x_2)$, $0 \le x_2 \le 1$ of Q. Corresponding terms with respect to the other three sides are indicated by +++. The other equivalent norm in (3.118) are covered by (1.309) and (1.312). As for the second equivalent quasi-norm in (3.117) we rely on Proposition 3.5. First we remark that the last expression in (3.117) can be written as in (1.295) (with $M = 2$). Then we apply (3.42) with $s = r$ to the corresponding quasi-norm for $B^r_{pp}(I)$ which shows that the continuous quasi-norm can be replaced by the discrete quasi-norm in the last equivalence in (3.42) with $q = p$. Now one can change the roles of x_1 and x_2. Using again (3.42) one obtains the equivalences in (3.117). □

Remark 3.15. With exception of the second equivalence in (3.117) all assertions of the above corollary are covered by preceding observations. Corresponding equivalent norms in terms of continuous differences in the spaces $S^r_{pp}B(Q)$ according to Theorem 1.67 are restricted by $p \ge 1$. This comes from a corresponding restriction in (1.166) for the spaces $S^r_{pq}B(\mathbb{R}^2)$ with a reference to [ST87, Theorem 2, p. 122]. In other words, (3.117) complements Theorem 1.67 (i) for some spaces $S^r_{pp}B(Q)$ with (3.115).

3.2.4 The spaces $S^r_{pq}B(\mathbb{Q}^2)$ and $S^r_pH(\mathbb{Q}^2)$

Let again

$$\mathbb{Q}^2 = \{x = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < 1\} \tag{3.119}$$

be the unit square in $\mathbb{R}^2$. In Theorem 3.13 we dealt with Faber bases in some spaces $S^r_{pp}B(\mathbb{Q}^2)$ and $S^1_pW(\mathbb{Q}^2)$ by specific arguments. The question arises whether these assertions can be extended to more general spaces $S^r_{pq}B(\mathbb{Q}^2)$ and $S^r_{pq}F(\mathbb{Q}^2)$ with the distinguished subclass $S^r_pH(\mathbb{Q}^2) = S^r_{p,2}F(\mathbb{Q}^2)$ of Sobolev spaces with dominating mixed smoothness. As before, $S^r_{pq}A(\mathbb{Q}^2)$ with $A \in \{B, F\}$ are the spaces with dominating mixed smoothness, introduced in Definition 1.56 as restriction of $S^r_{pq}A(\mathbb{R}^2)$ to $\mathbb{Q}^2$. Recall that

$$S^r_{pq}A(\mathbb{Q}^2) \hookrightarrow C(\mathbb{Q}^2) \quad \text{if } 0 < p, q \le \infty,\ r > 1/p, \tag{3.120}$$

with $p < \infty$ for F-spaces. This coincides with (3.87) where one finds references and comments. On the one hand we wish to extend Theorem 3.13 (i) from $S^r_{pp}B(\mathbb{Q}^2)$ to $S^r_{pq}B(\mathbb{Q}^2)$ with p, r as there and $0 < q \le \infty$. On the other hand we need for our later purposes some assertions about traces on $\partial\mathbb{Q}^2$ and equivalent quasi-norms of the same type as for $S^1_pW(\mathbb{Q}^2)$ in Proposition 1.68. As before we simplify our notation and abbreviate again $Q = \mathbb{Q}^2$ in what follows. For $0 < p, q \le \infty$, $r > 1/p$ we put

$$B^r_{pq}(\partial Q) = \big\{ f \in C(\partial Q) : f|I_j \in B^r_{pq}(I_j);\ j = 1,2,3,4 \big\} \tag{3.121}$$

with the same interpretation as in (1.306). Let as there

$$\mathrm{tr}_{\partial Q} : f \mapsto f|\partial Q \tag{3.122}$$

be the trace operator of spaces on $\mathbb{R}^2$ or on Q into spaces on ∂Q (if exists). From [ST87, Theorem 2.4.2, p. 133] it follows that

$$\mathrm{tr}_{\partial Q} : S^r_{pq}B(Q) \hookrightarrow B^r_{pq}(\partial Q) \tag{3.123}$$

is a linear and bounded map. One may ask whether there is a linear and bounded extension operator

$$\mathrm{ext}_{\partial Q} : B^r_{pq}(\partial Q) \hookrightarrow S^r_{pq}B(Q) \tag{3.124}$$

such that

$$\mathrm{tr}_{\partial Q} \circ \mathrm{ext}_{\partial Q} = \mathrm{id}, \quad \text{identity in } B^r_{pq}(\partial Q). \tag{3.125}$$

We are also interested in the existence of a linear and bounded extension operator

$$\mathrm{ext}_Q : S^r_{pq}B(Q) \hookrightarrow S^r_{pq}B(\mathbb{R}^2) \tag{3.126}$$

such that

$$\mathrm{re}_Q \circ \mathrm{ext}_Q = \mathrm{id}, \quad \text{identity in } S^r_{pq}B(Q). \tag{3.127}$$

Here $\mathrm{re}_Q f = f|Q$ is the restriction from $S^r_{pq}B(\mathbb{R}^2)$ onto $S^r_{pq}B(Q)$. In connection with (1.98)–(1.102) we explained what is meant by a common extension operator. Now we generalise Theorem 3.13 and extend Proposition 1.68 from $S^1_pW(Q)$ to some spaces $S^r_{pq}B(Q)$.

Theorem 3.16. *Let $Q = \mathbb{Q}^2$ be the square (3.119) and let $\{v_{km}\}$ be the Faber system (3.60)–(3.62). Let $d^2_{km}(f)$ be the mixed differences according to (3.74)–(3.77). Let $s^F_{pq}b(Q)$ be the sequence spaces as introduced in Definition 3.8. Let $S^r_{pq}B(Q)$ with*

$$0 < p < \infty, \quad \frac{1}{p} < r < 1 + \min\Big(\frac{1}{p}, 1\Big), \quad 0 < q < \infty, \tag{3.128}$$

be the above spaces, Figure 3.1, p. 127 (with r in place of s). Let $f \in D'(Q)$ (or likewise $f \in L_1(Q)$). Then $f \in S^r_{pq}B(Q)$ if, and only if, it can be represented as

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^F_k} \lambda_{km} \, 2^{-(k_1+k_2)(r-\frac{1}{p})} \, v_{km}, \quad \lambda \in s^F_{pq}b(Q), \tag{3.129}$$

unconditional convergence being in $L_1(Q)$. The representation (3.129) is unique with

$$\lambda_{km} = \lambda_{km}(f) = 2^{(k_1+k_2)(r-\frac{1}{p})} \, d^2_{km}(f), \quad k \in \mathbb{N}^2_{-1}, \quad m \in \mathbb{P}^F_k. \tag{3.130}$$

Furthermore,

$$J : f \mapsto \lambda(f) \tag{3.131}$$

is an isomorphic map of $S^r_{pq}B(Q)$ onto $s^F_{pq}b(Q)$, and $\{v_{km}\}$ is an unconditional basis in $S^r_{pq}B(Q)$. There exists a common extension operator ext_Q with (3.126), (3.127) and a common extension operator $\mathrm{ext}_{\partial Q}$ for the trace operator $\mathrm{tr}_{\partial Q}$ with (3.123)–(3.125). Furthermore,

$$\|f \,|S^r_{pq}B(Q)\|^* = \Big\| \frac{\partial^2 f}{\partial x_1 \partial x_2} \,|S^{r-1}_{pq}B(Q)\Big\| + \big\|\mathrm{tr}_{\partial Q} f \,|B^r_{pq}(\partial Q)\big\| \tag{3.132}$$

is an equivalent quasi-norm in $S^r_{pq}B(Q)$.

Proof. Step 1. Let f be given by (3.129). One obtains by (3.95) that this series converges absolutely in $C(Q)$ and hence also in $L_1(Q)$. Furthermore, $f \in C(Q)$. Then it follows from Theorem 3.10 that the representation is unique with $\lambda_{km}(f)$ as in (3.130). Let $\{v_0, v_1, v_{jm}\}$ be the Faber system (3.58), (3.59) on $I = (0,1)$. Let V_{jm} be the functions v_{jm} extended by zero from I to $\mathbb{R}$. Let

$$V_0(t) = \max(1 - |t|, 0), \quad V_1(t) = V_0(t-1), \quad t \in \mathbb{R}, \tag{3.133}$$

be the extension of v_0 and v_1 from I to $\mathbb{R}$. Let $V_{km}(x)$, $x \in \mathbb{R}^2$, be the corresponding product functions constructed in the same way as in (3.60). Now f given by (3.129) with $\lambda_{km} = \lambda_{km}(f)$ is extended from Q to $\mathbb{R}^2$ by

$$g = \mathrm{ext}_Q f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^F_k} \lambda_{km}(f) \, 2^{-(k_1+k_2)(r-\frac{1}{p})} \, V_{km}(x). \tag{3.134}$$

We wish to show that $g \in S^r_{pq}B(\mathbb{R}^2)$ and that for some $c > 0$ and all f,

$$\|g \,|S^r_{pq}B(\mathbb{R}^2)\| \le c \, \|\lambda(f) \,|s^F_{pq}b(Q)\|. \tag{3.135}$$

Recall that

$$\begin{aligned} \|g\,|S^r_{pq}B(\mathbb{R}^2)\| \sim \|g\,|S^{r-1}_{pq}B(\mathbb{R}^2)\| + \sum_{l=1}^{2} \Big\| \frac{\partial g}{\partial x_l}\,|S^{r-1}_{pq}B(\mathbb{R}^2)\Big\| \\ + \Big\| \frac{\partial^2 g}{\partial x_1 \partial x_2}\,|S^{r-1}_{pq}B(\mathbb{R}^2)\Big\| \end{aligned} \tag{3.136}$$

are equivalent quasi-norms, [ST87, Theorem 2, pp. 98/99]. Let χ_{km} with $k \in \mathbb{N}^2_{-1}$ and $m \in \mathbb{Z}^2$ be the characteristic functions of the rectangles Q_{km} in (2.236). It follows from (3.134) that

$$\frac{\partial^2 g}{\partial x_1 \partial x_2} = \sum_{k\in\mathbb{N}^2_{-1}} \sum_{m\in\mathbb{Z}^2} \mu_{km}(f)\, 2^{-(k_1+k_2)(r-1-\frac{1}{p})}\chi_{km} \tag{3.137}$$

where $\mu_{km}(f)$ are harmless linear combinations of neighbouring $\lambda_{k'm'}(f)$ (or zero). We assume temporarily $r > 1$. Then one can apply Proposition 2.34 with $r-1$ in place of r, hence

$$\Big\| \frac{\partial^2 g}{\partial x_1 \partial x_2}\,|S^{r-1}_{pq}B(\mathbb{R}^2)\Big\| \le c\,\|\lambda(f)\,|s^F_{pq}b(Q)\|. \tag{3.138}$$

According to Remark 2.36, Proposition 2.34 remains valid for hat functions V_{km} or products of hat functions and characteristic functions. This shows that there are counterparts of (3.138) with g and $\frac{\partial g}{\partial x_l}$ in place of $\frac{\partial^2 g}{\partial x_1 \partial x_2}$. Then (3.135) follows from (3.136) so far under the additional restriction $r > 1$. We extend (3.135) to the remaining admitted cases by complex interpolation

$$\big[S^{r_0}_{\infty\infty}B(\mathbb{R}^2),\, S^{r_1}_{p_1q_1}B(\mathbb{R}^2)\big]_\theta = S^r_{pq}B(\mathbb{R}^2), \quad 0<\theta<1, \tag{3.139}$$

$0 < r_0 < 1$,

$$0<p_1<\infty, \quad \max\Big(\frac{1}{p_1},1\Big) < r_1 < 1+\min\Big(\frac{1}{p_1},1\Big), \quad 0<q_1<\infty, \tag{3.140}$$

and

$$\frac{1}{p} = \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{\theta}{q_1}, \quad r = (1-\theta)r_0 + \theta r_1. \tag{3.141}$$

This is the extension of the complex interpolation method according to [MeM00], [KMM07], described in Theorem 1.22, from isotropic spaces to spaces with dominating mixed smoothness. We refer to [Vyb06, Theorem 4.6, p. 60]. From Theorem 3.13 (i) it follows that

$$\|g\,|S^{r_0}_{\infty\infty}B(\mathbb{R}^2)\| \le c\,\|\lambda(f)\,|s^F_{\infty\infty}b(Q)\|. \tag{3.142}$$

The interpolation (3.139) originates from a corresponding interpolation of respective sequence spaces. This is also covered by [Vyb06]. The interpolation property applied to the linear operator in (3.134) can be characterised in Figure 3.1, p. 127, by a line segment connecting $(0, r_0)$ and $\big(\frac{1}{p_1}, r_1\big)$. Together with $q_1 \to 0$ and $q_1 \to \infty$ one can reach any admitted space $S^r_{pq}B(\mathbb{R}^2)$. This proves (3.135) and by restriction $f \in S^r_{pq}B(Q)$ with

$$\|f\,|S^r_{pq}B(Q)\| \le c\,\|\lambda(f)\,|s^F_{pq}b(Q)\| \tag{3.143}$$

for all admitted cases.

Step 2. We prove the converse. Let $f \in S^r_{pq}B(Q)$ with p, q, r as in (3.128). By (3.120) and Theorem 3.10 or Theorem 3.13 applied to $S^r_{pq}B(Q) \hookrightarrow S^{r-\varepsilon}_{pp}B(Q)$, $0 < \varepsilon < r - \frac{1}{p}$, it follows that f can be represented by

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^F_k} \lambda_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})} v_{km}$$

with (3.130). We decompose f as in (3.97)–(3.100). Let $I_1 = (0, x_2)$, $0 \le x_2 \le 1$, be one of the four sides of Q. Then $f^{\partial Q}$ in (3.98) can be written as

$$f^{\partial Q} = v_0(x_1)\,\mathrm{tr}_{I_1} f + {+}{+} \tag{3.144}$$

where $+ + +$ refers to the three other sides of Q. Recall that

$$\mathrm{tr}_{I_1} f \in B^r_{pq}(I_1), \tag{3.145}$$

[ST87, Theorem 2.4.2, p. 133]. One can expand this function according to Theorem 3.1 (i) with $\mu_0 = \mu_1 = 0$ in (3.20). Then

$$f^{\partial Q}(x) = v_0(x_1) \sum_{k_2=0}^{\infty} \sum_{m_2=0}^{2^{k_2}-1} \lambda^1_{k_2 m_2}\, 2^{-(k_2-1)(r-\frac{1}{p})}\, v_{k_2 m_2}(x_2) + {+}{+} \tag{3.146}$$

and $\{\lambda^1_{k_2 m_2}\} \in b^+_{pq}(I_1)$ in the notation of Theorem 3.1. Similarly for the other three sides of Q and the harmless corner terms in $f^{\partial^2 Q}$. But this is the boundary part of $s^F_{pq}b(Q)$ in (3.66) with $k \in \mathbb{N}^2_{-1} \setminus \mathbb{N}^2_0$. By the arguments in Step 1 it follows that $f^{\partial^2 Q} + f^{\partial Q} \in S^r_{pq}B(Q)$. Hence

$$f^Q = \sum_{k \in \mathbb{N}^2_0} \sum_{m \in \mathbb{P}^F_k} \lambda_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})}\, v_{km} \in S^r_{pq}B(Q). \tag{3.147}$$

By (3.136) and its restriction to Q one has the expansion

$$\frac{\partial^2 f^Q}{\partial x_1 \partial x_2} = 4 \sum_{k \in \mathbb{N}^2_0} \sum_{m \in \mathbb{P}^F_k} \lambda_{km}\, 2^{-(k_1+k_2)(r-1-\frac{1}{p})} h_{km} \in S^{r-1}_{pq}B(Q) \tag{3.148}$$

according to Theorem 2.41 with $r-1$ in place of r in terms of the Haar tensor basis $\{h_{km}\}$ in (2.286). In particular,

$$\big\{\lambda_{km} : k \in \mathbb{N}^2_0,\ m \in \mathbb{P}^F_k = \mathbb{P}^H_k\big\} \in s^H_{pq}b(Q). \tag{3.149}$$

But this is the interior part of $s^F_{pq}b(Q)$ in (3.66). Then it follows that any $f \in S^r_{pq}B(Q)$ can be represented by (3.129). In particular, (3.129) is a one-to-one map of the quasi-Banach space $s^F_{pq}b(Q)$ onto the quasi-Banach space $S^r_{pq}B(Q)$. We justified in [T08, p. 199] that one can apply the Closed Graph Theorem as stated in [Woj91, pp. 3/4]. Hence (3.129) is an isomorphic map of $s^F_{pq}b(Q)$ onto $S^r_{pq}B(Q)$ and J in (3.131) is its inverse. In particular, $\{v_{km}\}$ is an unconditional basis.

Step 3. We prove the remaining assertions of the theorem. By the above considerations $f^{\partial Q}$ in (3.146) can be considered as an extension operator from $B^r_{pq}(I_j)$ with $j = 1,2,3,4$ to $S^r_{pq}B(\mathrm{Q})$. Complemented by a corresponding assertion for the corner-part $f^{\partial^2 Q}$ in (3.99) one obtains in this way a common extension operator

$$\mathrm{ext}_{\partial Q} \colon B^r_{pq}(\partial Q) \hookrightarrow S^r_{pq}B(Q), \quad \mathrm{tr}_{\partial Q} \circ \mathrm{ext}_{\partial Q} = \mathrm{id}, \tag{3.150}$$

identity in $B^r_{pq}(\partial Q)$. Furthermore, (3.134), (3.135) can now be interpreted as the construction of a common extension operator

$$\mathrm{ext}_Q \colon S^r_{pq}B(Q) \hookrightarrow S^r_{pq}B(\mathbb{R}^2), \quad \mathrm{re}_Q \circ \mathrm{ext}_Q = \mathrm{id}, \tag{3.151}$$

identity in $S^r_{pq}B(Q)$. It remains to prove (3.132). It follows from the above considerations that

$$\|f^Q \,|S^r_{pq}B(Q)\| \sim \left\| \frac{\partial^2 f^Q}{\partial x_1 \partial x_2} \,|S^{r-1}_{pq}B(Q)\right\| \tag{3.152}$$

and

$$\left\| \frac{\partial^2}{\partial x_1 \partial x_2} (f^{\partial Q} + f^{\partial^2 Q}) \,|S^{r-1}_{pq}B(Q)\right\| \le c \left\|\mathrm{tr}_{\partial Q} f \,|B^r_{pq}(\partial Q)\right\|. \tag{3.153}$$

Together with $f = f^Q + f^{\partial Q} + f^{\partial^2 Q}$ one obtains that

$$\begin{aligned} \|f \,|S^r_{pq}B(Q)\| &\le c\|f^Q \,|S^r_{pq}B(Q)\| + c\,\|\mathrm{tr}_{\partial Q} f \,|B^r_{pq}(\partial Q)\| \\ &\le c' \left\| \frac{\partial^2 f^Q}{\partial x_1 \partial x_2} \,|S^{r-1}_{pq}B(Q)\right\| + c'\,\|\mathrm{tr}_{\partial Q} f \,|B^r_{pq}(\partial Q)\| \\ &\le c'' \left\| \frac{\partial^2 f}{\partial x_1 \partial x_2} \,|S^{r-1}_{pq}B(Q)\right\| + c''\left\|\mathrm{tr}_{\partial Q} f \,|B^r_{pq}(\partial Q)\right\|. \end{aligned} \tag{3.154}$$

The converse follows from the above considerations. This proves that (3.132) is an equivalent quasi-norm in $S^r_{pq}B(Q)$. □

For our later use we need a modification of (3.152). For this purpose we introduce the following subspaces.

Definition 3.17. (i) Let $S^r_{pq}B(\mathbb{Q}^2)$ with

$$0 < p < \infty, \quad \frac{1}{p} < r < 1 + \min\left(\frac{1}{p}, 1\right), \quad 0 < q < \infty, \tag{3.155}$$

be the same spaces as in Theorem 3.16. Then

$$S^r_{pq}B(\mathbb{Q}^2)^{\urcorner} = \big\{ f \in S^r_{pq}B(\mathbb{Q}^2) : f(1,t) = f(t,1) = 0;\ 0 \le t \le 1 \big\}. \tag{3.156}$$

(ii) Let $S^1_p W(\mathbb{Q}^2)$ with $1 < p < \infty$ be the same Sobolev spaces with dominating mixed smoothness of first order as in Proposition 1.68, Theorem 3.13 (ii) and Corollary 3.14 (ii). Then

$$S^1_p W(\mathbb{Q}^2)^{\urcorner} = \big\{ f \in S^1_p W(\mathbb{Q}^2) : f(1,t) = f(t,1) = 0;\ 0 \le t \le 1 \big\}. \tag{3.157}$$

Remark 3.18. The definition makes sense. This follows for the spaces $S^r_{pq}B(\mathbb{Q}^2)$ from Theorem 3.16 and for the spaces $S^1_p W(\mathbb{Q}^2)$ from Proposition 1.68 or Corollary 3.14 (ii).

Corollary 3.19. (i) *Let p, q, r be as in (3.155). Then*

$$\frac{\partial^2}{\partial x_1 \partial x_2} : f \mapsto \frac{\partial^2 f}{\partial x_1 \partial x_2} \tag{3.158}$$

is an isomorphic map of $S^r_{pq}B(\mathbb{Q}^2)^{\urcorner}$ onto $S^{r-1}_{pq}B(\mathbb{Q}^2)$.

(ii) *Let $1 < p < \infty$. Then $\frac{\partial^2}{\partial x_1 \partial x_2}$ is an isomorphic map of $S^1_p W(\mathbb{Q}^2)^{\urcorner}$ onto $L_p(\mathbb{Q}^2)$.*

Proof. If $f \in S^r_{pq}B(Q)^{\urcorner}$, again with the abbreviation $Q = \mathbb{Q}^2$, then the representation (3.129) reduces to

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^H_k} \lambda_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})}\, v_{km}, \qquad \lambda \in s^H_{pq}b(Q), \tag{3.159}$$

with $\mathbb{P}^H_k$ as in (2.285) and $s^H_{pq}b(Q)$ according to (2.290). Instead of (3.148) one has

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 4 \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^H_k} \lambda'_{km}\, 2^{-(k_1+k_2)(r-1-\frac{1}{p})}\, h_{km}, \qquad \lambda' \in s^H_{pq}b(Q), \tag{3.160}$$

with $\lambda'_{km} = \lambda_{km}$ if $k \in \mathbb{N}^2_0$ and an immaterial modification if $k_1 = -1$ or $k_2 = -1$ (caused be different normalisations in (2.284) and (3.60), factors 2 and $\sqrt{2}$). We have (3.152) with $f \in S^r_{pq}B(Q)^{\urcorner}$ in place of f^Q. Then one obtains part (i) from Theorem 2.41 (i) with $r-1$ in place of r and Theorem 3.16. The corresponding assertion for $S^1_p W(Q)^{\urcorner}$ follows in the same way from Theorem 2.41 (ii) with $L_p(Q) = S^0_p H(Q)$ and Theorem 3.13 (ii). □

So far we have satisfactory expansions

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(f)\, v_{km} \tag{3.161}$$

in terms of Faber bases $\{v_{km}\}$ in $\mathbb{Q}^2$ for the spaces

$$S^r_{pq}B(\mathbb{Q}^2), \quad 0 < p < \infty, \ \tfrac{1}{p} < r < 1 + \min\big(\tfrac{1}{p}, 1\big), \quad 0 < q < \infty, \tag{3.162}$$

Theorem 3.16, Figure 3.1, p. 127, the spaces

$$C(\mathbb{Q}^2), \quad S^r C(\mathbb{Q}^2) = S^r_{\infty\infty}B(\mathbb{Q}^2), \quad 0 < r < 1, \tag{3.163}$$

Theorems 3.10, 3.13 (i), and the spaces

$$S^1_p W(\mathbb{Q}^2) = S^1_p H(\mathbb{Q}^2) = S^1_{p,2}F(\mathbb{Q}^2), \quad 1 < p < \infty, \tag{3.164}$$

Theorem 3.13 (ii). One may ask whether these assertions can be extended to other spaces $S^r_{pq}F(\mathbb{Q}^2)$, in particular to some (Hardy–)Sobolev spaces

$$S^r_p H(\mathbb{Q}^2) = S^r_{p,2}F(\mathbb{Q}^2) \tag{3.165}$$

with dominating mixed smoothness. According to Definition 1.56 the spaces $S^r_p H(\mathbb{Q}^2)$ are the restrictions of $S^r_p H(\mathbb{R}^2)$ to $\mathbb{Q}^2$. If $1 < p < \infty$ then one has the lifting (1.159) as a special case of (1.157), (1.158). In some sense $\frac{\partial^2}{\partial x_1 \partial x_2}$ in (3.158) is the adequate substitute of J_σ in (1.157) with $\sigma = 1$. The proofs of Theorem 3.16 and Corollary 3.19 rely, roughly speaking, on the observation that $\frac{\partial^2}{\partial x_1 \partial x_2}$ maps the Faber system $\{v_{km}\}$ onto the Haar tensor system $\{h_{km}\}$ in (2.286) which gives the possibility to employ Theorem 2.41. In case of the spaces $S^r_p H(\mathbb{Q}^2)$ we have the restriction (2.296), but based on the Remarks 2.39, 2.42 we conjectured that part (ii) of Theorem 2.41 (ii) can be extended to spaces $S^r_p H(\mathbb{Q}^2)$ with p, r as in (2.279). There are a few other points in the proof of Theorem 3.16 for which one has no immediate counterparts for the spaces $S^r_p H(\mathbb{Q}^2)$. This applies in particular to the use of complex interpolation of type (3.139). Nevertheless there are good reasons to assume that Theorem 3.16 and Corollary 3.19 can be complemented by corresponding assertions for the spaces $S^r_p H(\mathbb{Q}^2)$ with

$$\begin{cases} 2 \le p < \infty, & \tfrac{1}{2} < r < 1 + \tfrac{1}{p}, \\ 0 < p < 2, & \tfrac{1}{p} < r < \tfrac{3}{2}, \end{cases} \tag{3.166}$$

Figure 3.1, p. 127. Let

$$H^r_p(\partial\mathbb{Q}^2) = \big\{ f \in C(\partial\mathbb{Q}^2) : f|I_j \in H^r_p(I_j);\ j = 1,2,3,4 \big\} \tag{3.167}$$

be the counterpart of $B^r_{pq}(\partial\mathbb{Q}^2)$ in (3.121). With $\mathrm{tr}_{\partial\mathbb{Q}^2}$ as in (3.122), (3.123) one has now

$$\mathrm{tr}_{\partial\mathbb{Q}^2} : S^r_p H(\mathbb{Q}^2) \hookrightarrow H^r_p(\partial\mathbb{Q}^2), \tag{3.168}$$

[ST87, Theorem 2.4.2, p. 133]. Similarly as in (3.156) we put

$$S^r_p H(\mathbb{Q}^2)^{\urcorner} = \big\{ f \in S^r_p H(\mathbb{Q}^2) : f(1,t) = f(t,1) = 0;\ 0 \le t \le 1 \big\}. \tag{3.169}$$

We summarise the above comments as follows.

Conjecture 3.20. *Let $\mathbb{Q}^2$ be the square (3.119) and let $\{v_{km}\}$ be the Faber system (3.60)–(3.62). Let $d^2_{km}(f)$ be the mixed differences according to (3.74)–(3.77). Let $s^F_{pq} f(\mathbb{Q}^2)$ be the sequence spaces as introduced in Definition 3.8. Let $S^r_p H(\mathbb{Q}^2)$ be the above (Hardy–)Sobolev spaces with p, r as in (3.166), Figure 3.1, p. 127. Let $f \in D'(\mathbb{Q}^2)$ (or likewise $f \in L_1(\mathbb{Q}^2)$). Then $f \in S^r_p H(\mathbb{Q}^2)$ if, and only if, it can be uniquely represented as*

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^F_k} \lambda_{km} \, 2^{-(k_1+k_2)(r-\frac{1}{p})} \, v_{km}, \quad \lambda \in s^F_{p,2} f(\mathbb{Q}^2), \tag{3.170}$$

with $\lambda_{km} = \lambda_{km}(f)$ as in (3.130). Furthermore J in (3.131) is an isomorphic map of $S^r_p H(\mathbb{Q}^2)$ onto $s^F_{p,2} f(\mathbb{Q}^2)$ and $\{v_{km}\}$ is an unconditional basis in $S^r_p H(\mathbb{Q}^2)$. The common extension operators $\mathrm{ext}_{\partial\mathbb{Q}^2}$ and $\mathrm{ext}_{\mathbb{Q}^2}$ for the spaces covered by Theorem 3.16 are also related extension operators

$$\mathrm{ext}_{\partial\mathbb{Q}^2} : H^r_p(\partial\mathbb{Q}^2) \hookrightarrow S^r_p H(\mathbb{Q}^2) \tag{3.171}$$

and

$$\mathrm{ext}_{\mathbb{Q}^2} : S^r_p H(\mathbb{Q}^2) \hookrightarrow S^r_p H(\mathbb{R}^2) \tag{3.172}$$

for the above spaces $S^r_p H(\mathbb{Q}^2)$. Furthermore,

$$\| f \, | S^r_p H(\mathbb{Q}^2) \|^* = \Big\| \frac{\partial^2 f}{\partial x_1 \partial x_2} \big| S^{r-1}_p H(\mathbb{Q}^2) \Big\| + \big\| \mathrm{tr}_{\partial\mathbb{Q}^2} f \, | H^r_p(\partial\mathbb{Q}^2) \big\| \tag{3.173}$$

is an equivalent quasi-norm in $S^r_p H(\mathbb{Q}^2)$ and $\frac{\partial^2}{\partial x_1 \partial x_2}$ according to (3.158) is an isomorphic map of $S^r_p H(\mathbb{Q}^2)^{\urcorner}$ onto $S^{r-1}_p H(\mathbb{Q}^2)$.

Remark 3.21. The most important case $r = 1$, hence

$$S^1_p H(\mathbb{Q}^2) = S^1_p W(\mathbb{Q}^2), \quad 1 < p < \infty, \tag{3.174}$$

is covered by the above considerations. Otherwise there are good reasons supporting the above conjecture, but no final proofs. In case of the Besov spaces the situation is better, both by the above results, but also by the literature. As mentioned in Remark 3.7 there are a few papers, in particular [Cie77], [CKR93], dealing with Faber expansions and dyadic discrete spline characterisations for Besov spaces $B^s_{pq}(I)$ on the unit interval $I = (0,1)$. This has been extended in [Kam94, Theorem A, p. 173] to isotropic Besov spaces and anisotropic Besov spaces, including spaces of type $S^r_{pq} B(\mathbb{Q}^n)$, $1 \le p, q \le \infty$, $1/p < r < 1$. If $p = q$ then the corresponding assertions are closely related to (3.117). If $p \neq q$ then the respective spaces in [Kam94] are defined directly by (mixed) differences and not by restriction of related spaces on $\mathbb{R}^n$ (but this might be the same spaces, [Ull06]). Further discrete characterisations in terms of dyadic spline systems for Besov spaces may also be found in [Kam97] extending the result of the above-mentioned papers.

3.2.5 Higher dimensions

In Section 2.4.5 we dealt briefly with Haar tensor bases in n dimensions, $n \geq 3$. The extension of the corresponding assertions from two to higher dimensions caused no problems. One has the same situation in case of Faber bases. This may justify that we restrict ourselves to a few formulations. Let again

$$\mathbb{Q}^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_l < 1\} \tag{3.175}$$

be the unit cube in $\mathbb{R}^n$, $n \geq 2$. Let $S^r_{pq}B(\mathbb{Q}^n)$ and $S^r_{pq}F(\mathbb{Q}^n)$ with

$$S^r_p H(\mathbb{Q}^n) = S^r_{p,2}F(\mathbb{Q}^n), \quad 0 < p < \infty, \; r \in \mathbb{R}, \tag{3.176}$$

be the spaces according to Definition 1.56 with $\Omega = \mathbb{Q}^n$. Let

$$\{v_0, \; v_1, \; v_{jm} : j \in \mathbb{N}_0; \; m = 0, \ldots, 2^j - 1\} \tag{3.177}$$

be the Faber system (3.1)–(3.3) on the unit interval $I = (0, 1)$ of the real line $\mathbb{R}$. Let again $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$ and $\mathbb{N}^n_{-1}$ be the collection of all $k = (k_1, \ldots, k_n)$ with $k_j \in \mathbb{N}_{-1}$. Then there is an obvious generalisation of the Faber system in (3.60)–(3.62),

$$\{v_{km} : k \in \mathbb{N}^n_{-1}, \; m \in \mathbb{P}^{F,n}_k\}, \tag{3.178}$$

indicating now the dimension n, $\mathbb{P}^F_k = \mathbb{P}^{F,2}_k$. There are obvious generalisations $s^F_{pq}b(\mathbb{Q}^n)$ and $s^F_{pq}f(\mathbb{Q}^n)$ of the sequence spaces in Definition 3.8, where the additional terms in (3.67) must be taken over all faces of dimensions less than n, including edges and corners. Similarly the iterated mixed differences in (3.72)–(3.77) must be taken with respect to all axes of coordinates. In generalisation of Theorem 3.10 the Faber system $\{v_{km}\}$ in (3.178) is a (conditional) basis in $C(\mathbb{Q}^n)$. The Theorems 3.13, 3.16 and Corollary 3.14 have n-dimensional counterparts with $\mathbb{Q}^n$ in place of $\mathbb{Q}^2$ for the same parameters p, r, q as there, independent of n. In particular, the unique expansions for f are given now by

$$f = \sum_{k \in \mathbb{N}^n_{-1}} \sum_{m \in \mathbb{P}^{F,n}_k} \lambda_{km} \, 2^{-(k_1 + \cdots + k_n)(r - \frac{1}{p})} \, v_{km} \tag{3.179}$$

with

$$\lambda_{km} = \lambda_{km}(f) = 2^{(k_1 + \cdots + k_n)(r - \frac{1}{p})} \, d^2_{km}(f)^n, \quad k \in \mathbb{N}^n_{-1}, \; m \in \mathbb{P}^{F,n}_k, \tag{3.180}$$

where $d^2_{km}(f)^n$ refers to the n-dimensional version of (3.74)–(3.77). Also the assertions in Theorem 3.16 and Corollary 3.19 about traces, extension operators, equivalent quasi-norms and liftings have obvious n-dimensional counterparts. We fix a few of them for later purposes.

The boundary $\partial\mathbb{Q}^n$ of $\mathbb{Q}^n$ in (3.175) is the closure of the faces

$$\partial\mathbb{Q}^n_0 = \bigcup_{j=1}^n \mathbb{Q}^{n-1}_{0,j}, \quad \mathbb{Q}^{n-1}_{0,j} = \{x \in \mathbb{R}^n : x_j = 0; \; 0 < x_l < 1, \; l \neq j\} \tag{3.181}$$

anchored at $(0,\dots,0) \in \mathbb{R}^n$ and of the faces

$$\partial\mathbb{Q}^n_1 = \bigcup_{j=1}^n \mathbb{Q}^{n-1}_{1,j}, \quad \mathbb{Q}^{n-1}_{1,j} = \{x \in \mathbb{R}^n : x_j = 1;\ 0 < x_l < 1,\ l \neq j\} \tag{3.182}$$

anchored at $(1,\dots,1) \in \mathbb{R}^n$. Recall that in generalisation of (3.120),

$$S^r_{pq}B(\mathbb{Q}^n) \hookrightarrow C(\mathbb{Q}^n), \quad 0 < p,q \le \infty,\ r > 1/p. \tag{3.183}$$

In particular the n-dimensional version $\mathrm{tr}_{\partial\mathbb{Q}^n}$ of the trace in (3.122) makes sense,

$$\mathrm{tr}_{\partial\mathbb{Q}^n} : f \mapsto f|\partial\mathbb{Q}^n. \tag{3.184}$$

Instead of (3.121) we have now for $n \ge 3$,

$$S^r_{pq}B(\partial\mathbb{Q}^n) = \{f \in C(\partial\mathbb{Q}^n) : f|\mathbb{Q}^{n-1}_{l,j} \in S^r_{pq}B(\mathbb{Q}^{n-1}_{l,j});\ l = 0,1;\ j = 1,\dots,n\} \tag{3.185}$$

and, in generalisation of (3.123) and the arguments in [ST87, Theorem 2.4.2, p. 133],

$$\mathrm{tr}_{\partial\mathbb{Q}^n} : S^r_{pq}B(\mathbb{Q}^n) \hookrightarrow S^r_{pq}B(\partial\mathbb{Q}^n). \tag{3.186}$$

For p, r, q as in (3.155) we extend (3.156) to n dimensions,

$$S^r_{pq}B(\mathbb{Q}^n)^{\urcorner} = \{f \in S^r_{pq}B(\mathbb{Q}^n) : f|\partial\mathbb{Q}^n_1 = 0\} \tag{3.187}$$

where $f|\partial\mathbb{Q}^n_1$ is the trace of f on $\partial\mathbb{Q}^n_1$ according to (3.182). In addition to the above spaces $S^r_{pq}B(\mathbb{Q}^n)$ we are especially interested in the Sobolev spaces $S^1_pW(\mathbb{Q}^n)$ with $1 < p < \infty$. We refer to Section 1.2.5 where we introduced the underlying classical Sobolev spaces $S^1_pW(\mathbb{R}^n)$ which can be equivalently normed by (1.217). As above $S^1_pW(\mathbb{Q}^n)$ are the restrictions of $S^1_pW(\mathbb{R}^n)$ to $\mathbb{Q}^n$ according to Definition 1.56 with $\Omega = \mathbb{Q}^n$. They can be equivalently normed by

$$\|f\,|S^1_pW(\mathbb{Q}^n)\| \sim \sum_{\substack{\alpha\in\mathbb{N}^n_0 \\ \alpha_j\in\{0,1\}}} \|D^\alpha f\,|L_p(\mathbb{Q}^n)\|. \tag{3.188}$$

This is an extension of (1.297) and a consequence of the n-dimensional version of Theorem 1.67 (ii). One has (3.183) with $S^1_pW(\mathbb{Q}^n)$, $1 < p < \infty$, in place of $S^r_{pq}B(\mathbb{Q}^n)$. Hence, (3.184) makes sense and

$$\mathrm{tr}_{\partial\mathbb{Q}^n} : S^1_pW(\mathbb{Q}^n) \hookrightarrow S^1_pW(\partial\mathbb{Q}^n) \tag{3.189}$$

with

$$S^1_pW(\partial\mathbb{Q}^n) = \{f \in C(\partial\mathbb{Q}^n) : f|\mathbb{Q}^{n-1}_{l,j} \in S^1_pW(\mathbb{Q}^{n-1}_{l,j}),\ l = 0,1;\ j = 1,\dots,n\}. \tag{3.190}$$

We complement (3.187) by

$$S^1_pW(\mathbb{Q}^n)^{\urcorner} = \{f \in S^1_pW(\mathbb{Q}^n) : f|\partial\mathbb{Q}^n_1 = 0\}. \tag{3.191}$$

Proposition 3.22. *Let* $\mathbb{Q}^n$ *be the unit cube* (3.175), $n \geq 3$.

(i) *Let*

$$0 < p < \infty, \quad \frac{1}{p} < r < 1 + \min\left(\frac{1}{p}, 1\right), \quad 0 < q < \infty. \tag{3.192}$$

Then

$$\|f \,|S^r_{pq}B(\mathbb{Q}^n)\|^* = \left\| \frac{\partial^n f}{\partial x_1 \ldots \partial x_n} \,|S^{r-1}_{pq}B(\mathbb{Q}^n) \right\| + \left\| \mathrm{tr}_{\partial\mathbb{Q}^n} f \,|S^r_{pq}B(\partial\mathbb{Q}^n)\right\| \tag{3.193}$$

is an equivalent quasi-norm in $S^r_{pq}B(\mathbb{Q}^n)$. *Furthermore,*

$$\frac{\partial^n}{\partial x_1 \ldots \partial x_n} : f \mapsto \frac{\partial^n f}{\partial x_1 \ldots \partial x_n} \tag{3.194}$$

is an isomorphic map of $S^r_{pq}B(\mathbb{Q}^n)^{\urcorner}$ *onto* $S^{r-1}_{pq}B(\mathbb{Q}^n)$.

(ii) *Let* $1 < p < \infty$. *Then both* (3.188) *and*

$$\|f \,|S^1_pW(\mathbb{Q}^n)\|^* = \left\| \frac{\partial^n f}{\partial x_1 \ldots \partial x_n} \,|L_p(\mathbb{Q}^n) \right\| + \left\| \mathrm{tr}_{\partial\mathbb{Q}^n} f \,|S^1_pW(\partial\mathbb{Q}^n)\right\| \tag{3.195}$$

are equivalent norms in $S^1_pW(\mathbb{Q}^n)$. *Furthermore*

$$S^1_{p,\min(p,2)}B(\mathbb{Q}^n) \hookrightarrow S^1_pW(\mathbb{Q}^n) \hookrightarrow S^1_{p,\max(p,2)}B(\mathbb{Q}^n) \hookrightarrow C(\mathbb{Q}^n). \tag{3.196}$$

The operator $\frac{\partial^n}{\partial x_1 \ldots \partial x_n}$ *according to* (3.194) *is an isomorphic map of* $S^1_pW(\mathbb{Q}^n)^{\urcorner}$ *onto* $L_p(\mathbb{Q}^n)$.

Remark 3.23. These are n-dimensional versions of corresponding assertions for $n = 2$. We refer to Theorems 3.13, 3.16 and Corollaries 3.14, 3.19.

3.3 The spaces $S^r_{pq}\mathfrak{B}(\mathbb{Q}^2)$

3.3.1 Introduction and definition

It is one of the main aims of this book to study sampling numbers of compact embeddings between function spaces. Of special interest are embeddings of type

$$\mathrm{id}\colon B^s_{pq}(I) \hookrightarrow L_u(I), \quad \tfrac{1}{p} < s < 1 + \min\left(\tfrac{1}{p}, 1\right), \tag{3.197}$$

Figure 3.1, p. 127, where $I = (0,1)$ is the unit interval in $\mathbb{R}$ and

$$\mathrm{id}\colon S^r_{pq}B(Q) \hookrightarrow L_u(Q), \quad \tfrac{1}{p} < r < 1 + \min\left(\tfrac{1}{p}, 1\right), \tag{3.198}$$

where

$$Q = \mathbb{Q}^2 = \{x \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < 1\} \tag{3.199}$$

is the unit square in $\mathbb{R}^2$. Here $0 < u \le \infty$. If $u < 1$ then $L_u(I)$ and $L_u(Q)$ are no longer distributional spaces. Then it is reasonable to interpret id both in (3.197) and (3.198) in the topological context of spaces of measurable functions as described in 1.1.8. The otherwise somewhat isolated spaces $L_u(I)$ with $u < 1$ can be naturally incorporated in the larger class $\mathbf{A}^\sigma_{uv}(I)$ according to Definition 1.33 of possible target spaces in (3.197). Later on we deal with $L_u(I)$, suitable distributional spaces $A^\sigma_{uv}(I)$ and suitable spaces $\mathbf{A}^\sigma_{uv}(I)$ of measurable functions as target spaces. For spaces with dominating mixed smoothness the situation is less favourable. In particular there is no counterpart of the isotropic spaces $\mathbf{A}^s_{pq}(\mathbb{R}^n)$ and $\mathbf{A}^s_{pq}(\Omega)$ as considered in Section 1.1.8. It is not our aim to fill this gap. But we describe a possibility directly related to a possible replacement of $L_u(Q)$ in (3.198) by suitable spaces with dominating mixed smoothness of measurable functions. We take some aspects of the spaces $\mathbf{B}^s_{pq}(\mathbb{R}^n)$ and $\mathbf{B}^s_{pq}(\Omega)$ in Section 1.1.8 as a guide. As mentioned in Remark 1.37 with references to [Net89], [HeN07] and [T06, Section 9.2] these spaces can be characterised in terms of atomic decompositions. We are doing the same using the Faber system $\{v_{km}\}$ as atoms.

Let $\mathbf{M}(Q)$ be the collection of all equivalence classes of the Lebesgue almost everywhere finite complex-valued functions on the above square Q, furnished with the *convergence in measure* and converted into a complete metric space. One may consult Section 1.1.8 where we gave a more detailed description based on a reference to [Mall95, Section I,5] and where we also justified that the convergence in $L_p(Q)$, $0 < p \le \infty$, is stronger than the convergence in measure. One can take $\mathbf{M}(Q)$ as a substitute of $D'(Q)$ when dealing with spaces of measurable functions. We are interested only in some peculiar subspaces of $L_p(Q)$ what simplifies the situation somewhat.

Definition 3.24. Let $Q = \mathbb{Q}^2$ be the square (3.199) and let $\{v_{km}\}$ be the Faber system (3.60)–(3.62). Let $s^F_{pq}b(Q)$ be the sequence spaces as introduced in Definition 3.8. Let

$$0 < p \le \infty, \quad 0 < r < 1 + \min\left(\frac{1}{p}, 1\right), \quad 0 < q \le \infty, \tag{3.200}$$

Figure 3.2. Then $S^r_{pq}\mathfrak{B}(Q)$ is the collection of all $f \in \mathbf{M}(Q)$ which can be represented as

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^F_k} \lambda_{km} 2^{-(k_1+k_2)(r-\frac{1}{p})}\, v_{km}, \quad \lambda \in s^F_{pq}b(Q), \tag{3.201}$$

absolute convergence being in $L_p(Q)$. Furthermore,

$$\|f\,|S^r_{pq}\mathfrak{B}(Q)\| = \inf \|\lambda\,|s^F_{pq}b(Q)\|, \tag{3.202}$$

where the infimum is taken over all admissible representations (3.201).

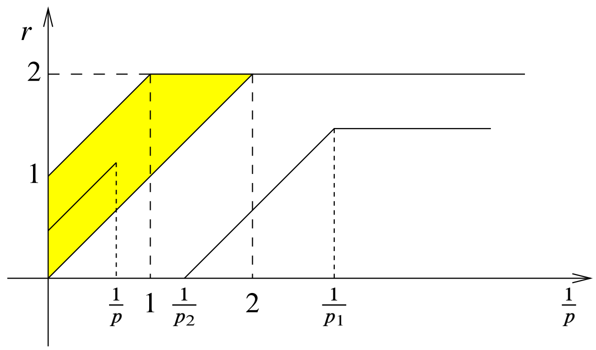

Figure 3.2. $S^r_{pq}\mathfrak{B}(Q)$ spaces.

Remark 3.25. We justify the absolute convergence of (3.201). If $1 \le p \le \infty$ then one obtains from (3.201) that

$$\begin{aligned} \|f \,|L_p(Q)\| &\le \sum_{k\in\mathbb{N}^2_{-1}} \Big(\sum_{m\in\mathbb{P}^F_k} |\lambda_{km}|^p \, 2^{-(k_1+k_2)(r-\frac{1}{p})p} \, 2^{-(k_1+k_2)} \Big)^{1/p} \\ &= \sum_{k\in\mathbb{N}^2_{-1}} 2^{-(k_1+k_2)r} \Big(\sum_{m\in\mathbb{P}^F_k} |\lambda_{km}|^p \Big)^{1/p} \end{aligned} \tag{3.203}$$

(modification if $p = \infty$). If $0 < p < 1$ then one has

$$\|f \,|L_p(Q)\| \le \Big(\sum_{k\in\mathbb{N}^2_{-1}} 2^{-(k_1+k_2)rp} \sum_{m\in\mathbb{P}^F_k} |\lambda_{km}|^p \Big)^{1/p}. \tag{3.204}$$

This shows that (3.201) converges absolutely in $L_p(Q)$ and that

$$\|f \,|L_p(Q)\| \le c\, \|f \,|S^r_{pq}\mathfrak{B}(Q)\|. \tag{3.205}$$

Formally one can extend the above definition to all $r > 0$. But this does not make much sense in our context. The restriction of r from above is suggested by (3.128) in Theorem 3.16. One may also compare the Figures 3.1, p. 127 and 3.2, p. 156. Let $\big(\frac{1}{p_1}, r\big)$ be an admitted couple and let $f \in S^r_{p_1 q}\mathfrak{B}(Q)$ be given by (3.201) with p_1 in place of p. Let $p_2 < p_1$. Then it follows from Hölder's inequality

$$2^{-\big(\frac{1}{p_2}-\frac{1}{p_1}\big)(k_1+k_2)} \Big(\sum_{m\in\mathbb{P}^F_k} |\lambda_{km}|^{p_2} \Big)^{1/p_2} \le \Big(\sum_{m\in\mathbb{P}^F_k} |\lambda_{km}|^{p_1} \Big)^{1/p_1}, \tag{3.206}$$

$k \in \mathbb{N}^2_0$, that f belongs also to $S^r_{p_2 q}\mathfrak{B}(Q)$ and that

$$\|f \,|S^r_{p_2 q}\mathfrak{B}(Q)\| \le c\, \|f \,|S^r_{p_1 q}\mathfrak{B}(Q)\|. \tag{3.207}$$

This again suggests to restrict r as in (3.200).

3.3.2 Properties

By (3.205) one has always

$$S^r_{pq}\mathfrak{B}(Q) \hookrightarrow L_p(Q), \quad 0 < p \le \infty, \tag{3.208}$$

for the spaces covered by Definition 3.24. It follows by standard arguments that $S^r_{pq}\mathfrak{B}(Q)$ is a quasi-Banach space. If $p \ge 1$, $q \ge 1$ then $s^F_{pq}b(Q)$, normed by (3.66), is a Banach space. Then $S^r_{pq}\mathfrak{B}(Q)$ with $p \ge 1$, $q \ge 1$ is also a Banach space. As before, $S^r_{pq}B(Q)$ are the (distributional) spaces with dominating mixed smoothness introduced in Definition 1.56 as restriction of $S^r_{pq}B(\mathbb{R}^2)$ to Q. One can compare these spaces with $S^r_{pq}\mathfrak{B}(Q)$ if one has both

$$S^r_{pq}B(Q) \hookrightarrow L_1(Q) \quad \text{and} \quad S^r_{pq}\mathfrak{B}(Q) \hookrightarrow L_1(Q). \tag{3.209}$$

This is especially the case if both spaces are continuously embedded in $C(Q)$.

Theorem 3.26. *Let $S^r_{pq}\mathfrak{B}(Q)$ be the spaces according to Definition 3.24.*

(i) *Let $0 < p_2 \le p_1 \le \infty$, $0 < q \le \infty$ and*

$$0 < r < 1 + \min\Big(\frac{1}{p_1}, 1\Big). \tag{3.210}$$

Then

$$S^r_{p_1 q}\mathfrak{B}(Q) \hookrightarrow S^r_{p_2 q}\mathfrak{B}(Q). \tag{3.211}$$

(ii) *Let $0 < p, q \le \infty$ and*

$$\frac{1}{p} < r < 1 + \min\Big(\frac{1}{p}, 1\Big), \tag{3.212}$$

Figures 3.1, p. 127, 3.2, p. 156. Then

$$S^r_{pq}\mathfrak{B}(Q) \hookrightarrow C(Q). \tag{3.213}$$

The representation (3.201) is unique with

$$\lambda_{km} = \lambda_{km}(f) = 2^{(k_1+k_2)(r-\frac{1}{p})}\, d^2_{km}(f), \quad k \in \mathbb{N}^2_{-1}, \quad m \in \mathbb{P}^F_k, \tag{3.214}$$

where $d^2_{km}(f)$ are the mixed differences according to (3.74)–(3.77). If, in addition, either $\max(p,q) < \infty$ or $p = q = \infty$ then

$$S^r_{pq}\mathfrak{B}(Q) = S^r_{pq}B(Q). \tag{3.215}$$

(iii) *Let $0 < p_1 < \infty$, $0 < r < 2$, $0 < q \le \infty$, and*

$$\frac{1}{p_2} = \frac{1}{p_1} - r \ge 0, \tag{3.216}$$

Figure 3.2, p. 156. Then

$$S^r_{p_1 q}\mathfrak{B}(Q) \hookrightarrow L_p(Q) \quad \textit{if } 0 < p < p_2. \tag{3.217}$$

If, in addition, $q \le \min(1, p_2)$ *then*

$$S^r_{p_1 q}\mathfrak{B}(Q) \hookrightarrow L_{p_2}(Q). \tag{3.218}$$

Proof. Step 1. Part (i) is covered by (3.207). We prove part (iii). Let $p_1 \le p \le p_2$. If $p \ge 1$ then one obtains from (3.201) with p_1 in place of p that

$$\|f\,|L_p(Q)\| \le \sum_{k\in\mathbb{N}^2_{-1}} 2^{-(k_1+k_2)(r-\frac{1}{p_1}+\frac{1}{p})}\Big(\sum_{m\in\mathbb{P}^F_k} |\lambda_{km}|^{p_1}\Big)^{1/p_1}. \tag{3.219}$$

If $p < p_2$ then one has

$$\|f\,|L_p(Q)\| \le c\,\|\lambda\,|s^F_{pq}b(Q)\| \tag{3.220}$$

for any q with $0 < q \le \infty$. If $p = p_2 \ge 1$ then one has (3.220) for all q with $0 < q \le 1$. This proves (3.217), (3.218) if $p \ge 1$. If $p < 1$ then one has to modify (3.219) as in (3.204). This proves part (iii).

Step 2. We prove part (ii). If $r > 1/p$ then one obtains (3.213) in the same way as in (3.95). This covers also that the representation (3.201) is unique with (3.214) = (3.90). Then (3.215) follows from Theorem 3.13 (i) if $p = q = \infty$, and from Theorem 3.16 if $p < \infty, q < \infty$. □

Remark 3.27. We compare the limiting embedding (3.218) with corresponding assertions for the classical spaces $B^r_{p_1 q}(I)$, Definition 1.24, and the spaces $\mathbf{B}^r_{p_1 q}(I)$, Definition 1.33. One has for $r > 0$,

$$B^r_{p_1 q}(I) \hookrightarrow L_{p_2}(I), \quad 0 < \tfrac{1}{p_2} = \tfrac{1}{p_1} - r < 1,\ q \le p_2, \tag{3.221}$$

[T01, p. 170], [T06, p. 40],

$$S^r_{p_1 q}B(Q) \hookrightarrow L_{p_2}(Q), \quad 0 < \tfrac{1}{p_2} = \tfrac{1}{p_1} - r < 1,\ q \le p_2, \tag{3.222}$$

[HaV09], and

$$\mathbf{B}^r_{p_1 q}(I) \hookrightarrow L_{p_2}(I), \quad 0 < \tfrac{1}{p_2} = \tfrac{1}{p_1} - r,\ q \le p_2, \tag{3.223}$$

[HaS08, Theorem 1.16]. Here (3.222) is a special case of the so-called Jawerth–Franke inequality using in addition (1.154) and its restriction to Q. This is also covered by [ScS04, Theorem 3] under the assumption $p_1 \ge 1$. These are the same conditions as in (3.218) if $p_2 \le 1$. But they are different if $p_2 > 1$. If $r > 1/p$ then one has (3.215). But it is doubtful whether this is also valid for other cases.

3.4 Bases in logarithmic spaces

3.4.1 Preliminaries

We dealt in Section 1.3 with logarithmic spaces. This will be complemented now by some assertions about Haar bases and Faber bases. But we are not interested in a systematic comprehensive theory. Just on the contrary, we restrict ourselves to those spaces which will be of some use later on and where we can rely on the techniques developed so far. Let $B^{s,b}_{pq}(\mathbb{R})$ be the spaces according to Definition 1.72 and let $B^{s,b}_{pq}(I)$ be the restriction of $B^{s,b}_{pq}(\mathbb{R})$ to the unit interval $I = (0,1)$ on the real line $\mathbb{R}$. These are special cases of the isotropic spaces $B^{s,b}_{pq}(\mathbb{R}^n)$ and $B^{s,b}_{pq}(\Omega)$ as introduced in Definition 1.77 and (1.347). However in higher dimensions we do not deal with isotropic spaces, but with the spaces $S^{r,b}_{pq}B(\mathbb{R}^2)$ and $S^{r,b}_{pq}B(\mathbb{Q}^2)$ according to Definitions 1.79, 1.81. We restrict ourselves again to the two-dimensional case and the unit square

$$Q = \mathbb{Q}^2 = \{x \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < 1\}. \tag{3.224}$$

We illustrate the necessary modifications when switching from spaces, say, $B^s_{pq}(\mathbb{R}^n) = B^{s,0}_{pq}(\mathbb{R}^n)$ to spaces $B^{s,b}_{pq}(\mathbb{R}^n)$ with $b \in \mathbb{R}$. In Definition 1.5 of atoms one has to replace the normalising factors in (1.31) and (1.32) by

$$|D^\alpha a_{jm}(x)| \le 2^{-j(s-\frac{n}{p})+j|\alpha|}(1+j)^{-b}, \quad |\alpha| \le K-1,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{3.225}$$

and

$$\begin{aligned} |D^\alpha a_{jm}(x) - D^\alpha a_{jm}(y)| \le 2^{-j(s-\frac{n}{p})+jK}(1+j)^{-b}|x-y|, \\ |\alpha| = K-1,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n. \end{aligned} \tag{3.226}$$

Then one has Theorem 1.7 (i) with $B^{s,b}_{pq}(\mathbb{R}^n)$ in place of $B^s_{pq}(\mathbb{R}^n)$ where the conditions for K, L, d and the sequence spaces b_{pq} are the same as there. This is covered by [Mou01, Theorem 1.18] and the above comments in Remark 1.6. But the additional factors $(1+j)^{-b}$ are typical and harmless in the above relevant estimates. There is a direct counterpart of Definition 1.9 of related kernels of local means and of Theorem 1.15 (i). This can be used to extend characterisations in terms of wavelets according to Theorem 1.18 (ii) from $B^s_{pq}(\mathbb{R}^n)$ to $B^{s,b}_{pq}(\mathbb{R}^n)$. But this follows also from real interpolation with function parameters as described in (1.351), (1.352) going back to [Mer84], [CoF88]. We refer also to [Alm05]. Afterwards one can argue as above. In the proof of the crucial Proposition 2.6 (i) (and in Remark 2.7) one has to replace the convergence-generating factors

$$2^{-\varepsilon|j-k|} \quad \text{by} \quad 2^{-\varepsilon|j-k|}\left(\frac{1+j}{1+k}\right)^b \le c\, 2^{-\delta|j-k|} \tag{3.227}$$

where $0 < \delta < \varepsilon, b \in \mathbb{R}, j \in \mathbb{N}_0, k \in \mathbb{N}_0$. There is a counterpart of Proposition 2.8 (i). This paves the way to extend the assertions for Haar bases and Faber bases from $B^s_{pq}(\mathbb{R})$, $B^s_{pq}(I)$ to $B^{s,b}_{pq}(\mathbb{R})$, $B^{s,b}_{pq}(I)$. A detailed formulation will be given in the next section.

We have a similar favourable situation for the spaces $S^{r,b}_{pq}B(\mathbb{R}^2)$, $S^{r,b}_{pq}B(\mathbb{Q}^2)$ compared with $S^r_{pq}B(\mathbb{R}^2)$, $S^r_{pq}B(\mathbb{Q}^2)$. With the same replacement as in (3.227), now in two dimensions, one has a counterpart of the crucial Proposition 2.34 (and Remark 2.36). Similarly there is a counterpart of Proposition 2.37. Both together pave the way to extend the assertions for Haar bases and Faber bases from $S^r_{pq}B(\mathbb{R}^2)$, $S^r_{pq}B(\mathbb{Q}^2)$ to $S^{r,b}_{pq}B(\mathbb{R}^2)$, $S^{r,b}_{pq}B(\mathbb{Q}^2)$. A detailed formulation will be given in Section 3.4.3 below.

3.4.2 The spaces $B^{s,b}_{pq}(\mathbb{R})$ and $B^{s,b}_{pq}(I)$

Let $B^{s,b}_{pq}(\mathbb{R})$ be the spaces introduced in Definition 1.72 and let $B^{s,b}_{pq}(I)$ be its restriction to the unit interval $I = (0,1)$ according to Definition 1.77 (ii). We collected in Section 1.3.2 a few properties. This will now be complemented by Haar bases and Faber bases relying on the hints in the preceding Section 3.4.1. First we extend Theorem 2.9 (i) using the same notation and conventions as there. In particular, $\{h_{jm}\}$ is the Haar system according to (2.93)–(2.96) and the sequence spaces b^-_{pq} have the same meaning as in (2.100), (2.101).

Theorem 3.28. *Let $0 < p \le \infty$, $0 < q \le \infty$,*

$$\frac{1}{p} - 1 < s < \min\Big(\frac{1}{p}, 1\Big), \tag{3.228}$$

and $b \in \mathbb{R}$. Let $f \in S'(\mathbb{R})$. Then $f \in B^{s,b}_{pq}(\mathbb{R})$ if, and only if, it can be represented as

$$f = \sum_{j=-1}^{\infty} \sum_{m\in\mathbb{Z}} \mu_{jm}\, 2^{-j(s-\frac{1}{p})}(2+j)^{-b}\, h_{jm}, \quad \mu \in b^-_{pq}, \tag{3.229}$$

unconditional convergence being in $S'(\mathbb{R})$ and locally in any space $B^\sigma_{pq}(\mathbb{R})$ with $\sigma < s$. The representation (3.229) is unique,

$$\mu_{jm} = \mu_{jm}(f) = 2^{j(s-\frac{1}{p}+1)}(2+j)^b \int_{\mathbb{R}} f(x)\, h_{jm}(x)\, \mathrm{d}x, \quad j \in \mathbb{N}_{-1},\ m \in \mathbb{Z}, \tag{3.230}$$

and

$$J:\ f \mapsto \mu(f) \tag{3.231}$$

is an isomorphic map of $B^{s,b}_{pq}(\mathbb{R})$ onto b^-_{pq}. If, in addition, $p < \infty$, $q < \infty$, then

$$\big\{2^{-j(s-\frac{1}{p})}(2+j)^{-b}\, h_{jm} :\ j \in \mathbb{N}_{-1},\ m \in \mathbb{Z}\big\} \tag{3.232}$$

is an unconditional (normalised) basis in $B^{s,b}_{pq}(\mathbb{R})$.

Remark 3.29. This is the generalisation of Theorem 2.9 (i) and the preceding technical explanations given there. We indicated in Section 3.4.1 that the corresponding proof can be carried over. This applies also to the additional use of duality and real interpolation. Another possibility would be to rely on the real interpolation (1.351) with functional parameter in the same way as in [Alm05].

Next we deal with Haar bases and Faber bases for the spaces $B^{s,b}_{pq}(I)$. This extends Theorem 2.13 (i) with s as in (2.134) from $B^s_{pq}(I)$ to $B^{s,b}_{pq}(I)$ and Theorem 3.1 (ii) with s as in (3.19) from $B^s_{pq}(I)$ to $B^{s,b}_{pq}(I)$. We use the same notation and technical explanations as there. In particular,

$$\{h_0,\ h_{jm} : j \in \mathbb{N}_0;\ m = 0,\dots,2^j - 1\} \tag{3.233}$$

is the same L_∞-normalised orthogonal Haar basis in $L_2(I)$ as in (2.128), (2.129). Let $b_{pq}(I)$ be the related sequence spaces according to (2.130), (2.131). Let

$$\{v_0,\ v_1,\ v_{jm} : j \in \mathbb{N}_0;\ m = 0,\dots,2^j - 1\} \tag{3.234}$$

be the Faber system (3.1)–(3.3). Let $b^+_{pq}(I)$ be the related sequence spaces according to (3.16), (3.17). Let $(\Delta^2_{2^{-j-1}} f)(2^{-j}m)$ be the same second differences as in (3.5).

Theorem 3.30. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$,

$$\frac{1}{p} - 1 < s < \min\Big(\frac{1}{p}, 1\Big), \tag{3.235}$$

and $b \in \mathbb{R}$, *Figure 2.3, p. 82. Let* $f \in D'(I)$. *Then* $f \in B^{s,b}_{pq}(I)$ *if, and only if, it can be represented as*

$$f = \mu_0\, h_0 + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} \mu_{jm}\, 2^{-j(s-\frac{1}{p})} (1+j)^{-b}\, h_{jm}, \quad \mu \in b_{pq}(I), \tag{3.236}$$

unconditional convergence being in $B^\sigma_{pq}(I)$ *with* $\sigma < s$. *The representation (3.236) is unique,*

$$\begin{cases} \mu_0 = \mu_0(f) = \int_I f(x)\, h_0(x)\,\mathrm{d}x, \\ \mu_{jm} = \mu_{jm}(f) = 2^{j(s-\frac{1}{p}+1)} (1+j)^b \int_I f(x)\, h_{jm}(x)\,\mathrm{d}x, \end{cases} \tag{3.237}$$

$j \in \mathbb{N}_0$; $m = 0,\dots,2^j - 1$. *Furthermore,*

$$J : f \mapsto \mu(f) \tag{3.238}$$

is an isomorphic map of $B^{s,b}_{pq}(I)$ *onto* $b_{pq}(I)$. *If, in addition,* $p < \infty$, $q < \infty$, *then* (3.233) *is an unconditional basis in* $B^{s,b}_{pq}(I)$.

(ii) *Let* $0 < p \le \infty$, $0 < q \le \infty$,

$$\frac{1}{p} < s < 1 + \min\Big(\frac{1}{p}, 1\Big), \tag{3.239}$$

and $b \in \mathbb{R}$, Figure 3.1, p. 127. Let $f \in D'(I)$. Then $f \in B^{s,b}_{pq}(I)$ if, and only if, it can be represented as

$$f = \mu_0 v_0 + \mu_1 v_1 + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} \mu_{jm} 2^{-j(s-\frac{1}{p})} (1+j)^{-b} v_{jm}, \quad \mu \in b^+_{pq}(I), \tag{3.240}$$

unconditional convergence being in $B^{\sigma}_{pq}(I)$ with $\sigma < s$ and in $C(I)$. The representation (3.240) is unique, $\mu = \mu(f)$ with

$$\begin{aligned} \mu_0(f) = f(0), \quad \mu_1(f) = f(1), \\ \mu_{jm}(f) = -2^{j(s-\frac{1}{p})-1}(1+j)^b \big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m) \end{aligned} \tag{3.241}$$

where $j \in \mathbb{N}_0$ and $m = 0, \ldots, 2^j - 1$. Furthermore,

$$J : f \mapsto \mu(f) \tag{3.242}$$

is an isomorphic map of $B^{s,b}_{pq}(I)$ onto $b^+_{pq}(I)$. If, in addition, $p < \infty$, $q < \infty$, then (3.234) is an unconditional basis in $B^{s,b}_{pq}(I)$.

Remark 3.31. This is the generalisation of Theorem 2.13 (i) and Theorem 3.1 (i). We took over the corresponding formulations and notation. Otherwise we outlined in Section 3.4.1 that the key ingredients of the proofs remain valid if switching from $B^s_{pq}(I)$ to $B^{s,b}_{pq}(I)$. But we add a few comments. Recall that

$$B^{s-\varepsilon}_{pq}(I) \hookrightarrow B^{s,b}_{pq}(I) \hookrightarrow B^{s+\varepsilon}_{pq}(I) \quad \text{for any } b \in \mathbb{R} \text{ and } \varepsilon > 0. \tag{3.243}$$

The reduction of part (i) to Theorem 3.28 can be done similarly as in the proof of Theorem 2.13 where (3.243) ensures that $B^{s,b}_{pq}(I)$ can be identified with the counterpart of (2.141). The pointwise multiplier property of characteristic functions follows also from the interpolation (1.352). In case of part (ii) one can follow the proof of Theorem 3.1 combined with the corresponding comments in Section 3.4.1. The substitute of (3.30) follows from the above observation that v_{jm} are (not normalised) atoms in the counterpart of (3.29). In connection with (3.32)–(3.34) we used (3.6), (3.7). But this remains also valid for the spaces $B^{s,b}_{pq}(I)$. This can be justified again by a reference to [T08, Proposition 4.21, p. 113] combined with (1.353)–(1.355). This proves part (ii).

Remark 3.32. One may ask for equivalent quasi-norms for the above spaces $B^{s,b}_{pq}(\mathbb{R})$ and $B^{s,b}_{pq}(I)$. So far we have Proposition 1.74 based on the references in Remark 1.75 for the spaces $B^{s,b}_{pq}(\mathbb{R})$ and related comments (or conjectures) for the spaces $B^{s,b}_{pq}(I)$ in Remark 1.76. In Proposition 3.5 we collected equivalent quasi-norms for the spaces $B^s_{pq}(I)$ with (3.239)=(3.40). There is hardly any doubt that there is a full counterpart of (3.42) for the spaces $B^{s,b}_{pq}(I)$. But this is not covered by the previous considerations. So we restrict ourselves to the extension of the last expression in (3.42) from $B^s_{pq}(I)$ to $B^{s,b}_{pq}(I)$:

Let $B^{s,b}_{pq}(I)$ be the same spaces as in Theorem 3.30 (ii) *with $0 < p, q \le \infty$, $b \in \mathbb{R}$ and*

$$\frac{1}{p} < s < 1 + \min\left(\frac{1}{p}, 1\right). \tag{3.244}$$

Then $f \in B^{s,b}_{pq}(I)$ can be represented as

$$f(x) = f(0)(1-x) + f(1)x - \frac{1}{2}\sum_{j=0}^{\infty}\sum_{m=0}^{2^j-1} \big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m)\, v_{jm}(x), \tag{3.245}$$

$x \in I$, and

$$\begin{aligned} &\|f \,|B^{s,b}_{pq}(I)\| \\ &\quad \sim |f(0)| + |f(1)| \\ &\qquad + \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{1}{p})q}(1+j)^{bq}\Big(\sum_{m=0}^{2^j-1} \big|\big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m)\big|^p\Big)^{q/p}\Big)^{1/q} \end{aligned} \tag{3.246}$$

is an equivalent quasi-norm (*with the usual modification if $p = \infty$ and/or $q = \infty$*).

3.4.3 The spaces $S^{r,b}_{pq}B(\mathbb{R}^2)$ and $S^{r,b}_{pq}B(\mathbb{Q}^2)$

In Section 3.4.2 we extended Haar bases and Faber bases for $B^s_{pq}(\mathbb{R})$, $B^s_{pq}(I)$ to their logarithmic generalisations $B^{s,b}_{pq}(\mathbb{R})$, $B^{s,b}_{pq}(I)$. Now we are doing the same for spaces with dominating mixed smoothness extending Haar bases and Faber bases for $S^r_{pq}B(\mathbb{R}^2)$, $S^r_{pq}B(\mathbb{Q}^2)$ to their logarithmic generalisations $S^{r,b}_{pq}B(\mathbb{R}^2)$, $S^{r,b}_{pq}B(\mathbb{Q}^2)$. Recall that $\mathbb{Q}^2$ is the unit square (3.224) in $\mathbb{R}^2$. Let $S^{r,b}_{pq}B(\mathbb{R}^2)$ and $S^{r,b}_{pq}B(\mathbb{Q}^2)$ be the spaces introduced in the Definitions 1.79, 1.81. First we extend Theorem 2.38 using the same notation and conventions as there. In particular,

$$\big\{h_{km} : k \in \mathbb{N}^2_{-1},\ m \in \mathbb{Z}^2\big\} \tag{3.247}$$

is the (L_∞-normalised) Haar tensor system (2.225), (2.226) with $n = 2$ based on (2.224). Let $s_{pq}b(\mathbb{R}^2)$ be the sequence spaces introduced in Definition 2.30.

Theorem 3.33. *Let $0 < p \le \infty$, $0 < q \le \infty$ ($1 < q \le \infty$ if $p = \infty$), $b \in \mathbb{R}$ and*

$$\frac{1}{p} - 1 < r < \min\left(\frac{1}{p}, 1\right), \tag{3.248}$$

Figure 2.3, p. 82 (*with $s = r$*). *Let $f \in S'(\mathbb{R}^2)$. Then $f \in S^{r,b}_{pq}B(\mathbb{R}^2)$ if, and only if, it can be represented as*

$$f = \sum_{k\in\mathbb{N}^2_{-1}}\sum_{m\in\mathbb{Z}^2} \mu_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})}(2+k_1)^{-b}(2+k_2)^{-b} h_{km}, \quad \mu \in s_{pq}b(\mathbb{R}^2), \tag{3.249}$$

unconditional convergence being in $S'(\mathbb{R}^2)$ *and locally in any space* $S^{\varrho}_{pq}B(\mathbb{R}^2)$ *with* $\varrho < r$. *The representation* (3.249) *is unique,*

$$\mu_{km} = \mu_{km}(f) = 2^{(k_1+k_2)(r-\frac{1}{p}+1)}(2+k_1)^b(2+k_2)^b \int_{\mathbb{R}^2} f(x)\, h_{km}(x)\,\mathrm{d}x. \tag{3.250}$$

Furthermore,

$$J : f \mapsto \{\mu_{km}(f)\} \tag{3.251}$$

is an isomorphic map of $S^{r,b}_{pq}B(\mathbb{R}^2)$ *onto* $s_{pq}b(\mathbb{R}^2)$. *If, in addition,* $p < \infty$, $q < \infty$, *then* (3.247) *is an unconditional basis in* $S^{r,b}_{pq}B(\mathbb{R}^2)$.

Remark 3.34. This is the direct generalisation of Theorem 2.38. The most substantial ingredients of the related proof are the Propositions 2.34, 2.37. They can be extended to the above situation as mentioned at the end of Section 3.4.1. This applies also to the other tools used in the proof of Theorem 2.38 such as duality and interpolation.

Next we deal with Haar bases and Faber bases for the spaces $S^{r,b}_{pq}B(\mathbb{Q}^2)$. This extends Theorem 2.41 (i) and the related parts of Theorem 3.16. We use the same notation and technical explanations as there. In particular,

$$\{h_{km} : k \in \mathbb{N}^2_{-1},\ m \in \mathbb{P}^H_k\} \tag{3.252}$$

is the Haar tensor system (2.284)–(2.286). Let $s^H_{pq}b(\mathbb{Q}^2)$ be the related sequence spaces according to (2.289), (2.290). Let

$$\{v_{km} : k \in \mathbb{N}^2_{-1},\ m \in \mathbb{P}^F_k\} \tag{3.253}$$

be the Faber system (3.60)–(3.62). Let $d^2_{km}(f)$ be the mixed differences according to (3.74)–(3.77). Let $s^F_{pq}b(\mathbb{Q}^2)$ be the sequence spaces as introduced in Definition 3.8.

Theorem 3.35. (i) *Let* $0 < p \le \infty$, $0 < q \le \infty$ ($1 < q \le \infty$ *if* $p = \infty$), $b \in \mathbb{R}$, *and*

$$\frac{1}{p} - 1 < r < \min\Big(\frac{1}{p}, 1\Big), \tag{3.254}$$

Figure 2.3, *p.* 82 (*with* r *in place of* s). *Let* $f \in D'(\mathbb{Q}^2)$. *Then* $f \in S^{r,b}_{pq}B(\mathbb{Q}^2)$ *if, and only if, it can be represented as*

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^H_k} \lambda_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})}(2+k_1)^{-b}(2+k_2)^{-b}\, h_{km}, \quad \lambda \in s^H_{pq}b(\mathbb{Q}^2), \tag{3.255}$$

unconditional convergence being in $D'(\mathbb{Q}^2)$ *and in any space* $S^{\varrho}_{pq}B(\mathbb{Q}^2)$ *with* $\varrho < r$. *The representation* (3.255) *is unique,* $\lambda = \lambda(f)$,

$$\lambda_{km} = \lambda_{km}(f) = 2^{(k_1+k_2)(r-\frac{1}{p})}(2+k_1)^b(2+k_2)^b \int_{\mathbb{Q}^2} f(x)\, h_{km}(x)\,\mathrm{d}x. \tag{3.256}$$

Furthermore,

$$J : f \mapsto \lambda(f) \tag{3.257}$$

is an isomorphic map of $S^{r,b}_{pq}B(\mathbb{Q}^2)$ *onto* $s^H_{pq}b(\mathbb{Q}^2)$. *If, in addition,* $p < \infty$, $q < \infty$, *then* (3.252) *is an unconditional basis in* $S^{r,b}_{pq}(\mathbb{Q}^2)$.

(ii) *Let*

$$0 < p < \infty, \quad \frac{1}{p} < r < 1 + \min\left(\frac{1}{p}, 1\right), \quad 0 < q < \infty, \tag{3.258}$$

and $b \in \mathbb{R}$, *Figure* 3.1, *p.* 127 (*with* r *in place of* s). *Let* $f \in D'(\mathbb{Q}^2)$ (*or likewise* $f \in L_1(\mathbb{Q}^2)$). *Then* $f \in S^{r,b}_{pq}B(\mathbb{Q}^2)$ *if, and only if, it can be represented as*

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^F_k} \lambda_{km}\, 2^{-(k_1+k_2)(r-\frac{1}{p})}(2+k_1)^{-b}(2+k_2)^{-b}\, v_{km}, \quad \lambda \in s^F_{pq}b(\mathbb{Q}^2), \tag{3.259}$$

unconditional convergence being in $L_1(\mathbb{Q}^2)$. *The representation* (3.259) *is unique,* $\lambda = \lambda(f)$,

$$\lambda_{km} = \lambda_{km}(f) = 2^{(k_1+k_2)(r-\frac{1}{p})}(2+k_1)^{b}(2+k_2)^{b}\, d^2_{km}(f), \quad k \in \mathbb{N}^2_{-1},\ m \in \mathbb{P}^F_k. \tag{3.260}$$

Furthermore,

$$J : f \mapsto \lambda(f) \tag{3.261}$$

is an isomorphic map of $S^{r,b}_{pq}B(\mathbb{Q}^2)$ *onto* $s^F_{pq}b(\mathbb{Q}^2)$ *and* (3.253) *is an unconditional basis in* $S^{r,b}_{pq}B(\mathbb{Q}^2)$.

Remark 3.36. Part (i) is the extension of Theorem 2.41 (i). It follows from the above Theorem 3.33 in the same way as in the proof of Theorem 2.41. Similarly, part (ii) extends some assertions from Theorem 3.16. We gave a detailed proof of this theorem based on preceding observations and some references. All this can be extended to the above case. However we must admit that this has not yet been done for all ingredients needed. But this does not matter very much for what follows. We do not need that $S^{r,b}_{pq}B(\mathbb{Q}^2)$ is the restriction of $S^{r,b}_{pq}B(\mathbb{R}^2)$ to $\mathbb{Q}^2$, but that it is the collection of all f which can be represented by (3.259). Then one has also (3.260) and the isomorphic map (3.261). The other assertions in Theorem 3.16 can also be carried over.

3.5 Faber splines: an outlook

3.5.1 Preparations

The restriction of the Haar system $\{h_{jm} : j \in \mathbb{N}_{-1}, m \in \mathbb{Z}\}$ on $\mathbb{R}$ according to (2.96) to the unit interval $I = (0,1)$ on $\mathbb{R}$ causes no problem. One obtains the Haar system

(2.128), (2.129). Integration of h_{jm} gives essentially the related Faber functions in (2.5), (3.3),

$$v_{jm}(x) = 2^{j+1} \int_{-\infty}^{x} h_{jm}(y)\,\mathrm{d}y, \quad x \in \mathbb{R},\ j \in \mathbb{N}_0;\ m = 0, \ldots, 2^j - 1. \tag{3.262}$$

This simple observation explains why the Faber expansions in Theorem 3.1 can be reduced to corresponding assertions for Haar bases, where the smoothness $s-1$ for Haar bases is lifted by 1, hence s with (3.19) for B-spaces, Figure 3.1, p. 127. Afterwards we extended these assertions to higher dimensions and related spaces with dominating mixed smoothness. This will be fundamental when dealing in the following chapters with sampling, numerical integration and discrepancy. For pointwise evaluation of, say, $f \in B^s_{pq}(I)$ the restriction $s > 1/p$ is natural. But the restriction $s < 1 + \min(\frac{1}{p}, 1)$ in (3.19) originates from the limited smoothness of the hat-functions v_{jm}. In this Section 3.5 we outline a way how to overcome this restriction. Basically we replace the Haar functions h_{jm} by the spline functions h^l_{jm} as considered in Section 2.5.2, especially Theorem 2.49. However this causes some problems, and a detailed theory might be a rather substantial undertaking. We restrict ourselves to a few key ideas which show that there is a way to extend the above theory (and also what follows) from, say, spaces $B^s_{pq}(I)$ with p, q, s as in Theorem 3.1 to spaces $B^s_{pq}(I)$ with

$$0 < p, q \le \infty, \quad \frac{1}{p} < s < 2l + 1 + \min\Big(\frac{1}{p}, 1\Big), \quad l \in \mathbb{N}_0. \tag{3.263}$$

We begin with two preparations. First we show that it is sufficient to deal with functions, say, $f \in B^s_{pq}(\mathbb{R})$ with compact support in a fixed interval. Secondly we wish to provide a better understanding of what follows. For this purpose we sketch typical arguments used later on in case of Haar functions and Faber functions where we already know the outcome.

Preparation 1. Let again $I = (0, 1)$ be the unit interval on $\mathbb{R}$ and let

$$f \in B^s_{pq}(I), \quad 0 < p, q \le \infty,\ s > \max\big(\tfrac{1}{p}, 1\big) - 1. \tag{3.264}$$

Then f can be extended from I to $\mathbb{R}$ by the classical Hestenes method. We may assume that $\operatorname{supp} f \subset [0, 1/2)$. Then we extend f from $x > 0$ to $x < 0$ by

$$\mathrm{ext}_L f(x) = \begin{cases} f(x) & \text{if } x > 0, \\ \sum_{l=1}^{L} a_l f(-lx) & \text{if } x < 0, \end{cases} \tag{3.265}$$

with $s < L \in \mathbb{N}$ and

$$\sum_{l=1}^{L} (-1)^k a_l l^k = 1, \quad k = 0, \ldots, L - 1. \tag{3.266}$$

We refer for details to [T92, Section 4.5.2, p. 223] (and to [HaT08, Section 4.6.1, p. 112] for historical comments). If, in addition, $s > 1/p$ then pointwise evaluation makes sense and

$$\mathrm{ext}_L f(2^{-j}m) = \sum_{l=1}^{L} a_l\, f\big(-2^{-j}ml\big), \quad j \in \mathbb{N}_0,\ 0 > m \in \mathbb{Z}. \tag{3.267}$$

In particular the function values of $\mathrm{ext}_L f$ in the lattice points $2^{-j}\mathbb{Z}$ can be reduced to function values in $I \cap 2^{-j}\mathbb{Z}$. Later on we use expansions of type (3.20), (3.21) for estimates of sampling numbers from above. One may consult Section 4.2 below. Then it is quite clear by the above observation (3.267) that any such estimate for $\mathrm{ext}_L f$ gives also a corresponding estimate for $f \in B^s_{pq}(I)$. In other words, it is sufficient to deal with functions $f \in B^s_{pq}(\mathbb{R})$ with $\operatorname{supp} f \subset I$ avoiding discussions what happens at the endpoints 0 and 1 of I.

Preparation 2. We outline how the above Faber expansions can be obtained from Haar expansions by direct arguments. This will be taken as a guide for what follows later on. Let

$$f \in B^s_{pq}(\mathbb{R}) \quad \text{with } \operatorname{supp} f \subset I, \tag{3.268}$$

and

$$0 < p,q \le \infty, \quad \frac{1}{p} < s < 1 + \min\Big(\frac{1}{p}, 1\Big). \tag{3.269}$$

We apply Theorem 2.9 to $f' \in B^{s-1}_{pq}(\mathbb{R})$ where $\{h_{jm}\}$ is the Haar system (2.93)–(2.96). Let χ_m be the characteristic function of the interval $(m, m+1)$ with $m \in \mathbb{Z}$ and $\big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m)$ be the second differences according to (3.5). Then one has

$$\begin{aligned} f'(x) &= \sum_{m\in\mathbb{Z}} \chi_m(x) \int_{\mathbb{R}} f'(y)\,\chi_m(y)\,\mathrm{d}y + \sum_{j\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}} 2^j h_{jm}(x) \int_{\mathbb{R}} f'(y) h_{jm}(y)\,\mathrm{d}y \\ &= -\sum_{j\in\mathbb{N}_0}\sum_{m\in\mathbb{Z}} 2^j \big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m)\, h_{jm}(x). \end{aligned} \tag{3.270}$$

Integration gives

$$f(x) = \int_{-\infty}^{x} f'(y)\,\mathrm{d}y = -\frac{1}{2} \sum_{j\in\mathbb{N}_0} \sum_{m=0}^{2^j-1} \big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m)\, v_{jm}(x), \quad x \in \mathbb{R}, \tag{3.271}$$

in terms of the Faber functions in (3.3). This coincides with (3.4) as $f(0) = f(1) = 0$, extended from I to $\mathbb{R}$.

3.5.2 Faber splines

Let $h^l(x) = h^l_M(x)$ be the basic spline of order l, $l \in \mathbb{N}_0$, according to Section 2.5.1 with the properties mentioned there. If $l = 0$ then we assume that $h^0(x)$ coincides with (2.93). In particular,

$$v^0(x) = v_{0,0}(x) = 2 \int_{-\infty}^{x} h^0(y)\,\mathrm{d}y, \quad x \in \mathbb{R}, \tag{3.272}$$

is the basic Faber function in (3.3) extended from I to $\mathbb{R}$ by zero. We call

$$v^l(x) = 2 \int_{-\infty}^{x} \int_{-\infty}^{t_l} \dots \int_{-\infty}^{t_1} h^l(t)\,\mathrm{d}t\,\mathrm{d}t_1 \dots \mathrm{d}t_l, \quad x \in \mathbb{R}, \tag{3.273}$$

a *basic Faber spline* (of order l), where $l \in \mathbb{N}_0$ (with v^0 as in (3.272)). We collect some properties which originate from the properties (i)–(iv) of h^l in Section 2.5.1.

Proposition 3.37. *Let $l \in \mathbb{N}_0$ and let v^l be the above functions.*

(i) *Then v^l has classical continuous derivatives up to order $2l$ inclusively on $\mathbb{R}$.*

(ii) *The restriction of v^l to each interval $(m, m + \frac{1}{2})$ with $2m \in \mathbb{Z}$ is a polynomial of degree at most $2l + 1$.*

(iii) *There are constants $c > 0$, $\alpha > 0$, such that*

$$\left| \frac{\mathrm{d}^k}{\mathrm{d}x^k} v^l(x) \right| \le c\, e^{-\alpha |x|}, \quad 2x \in \mathbb{R} \setminus \mathbb{Z}, \tag{3.274}$$

and $k = 0, \dots, 2l + 1$.

(iv) *Furthermore,*

$$\int_{\mathbb{R}} v^l(x)\,\mathrm{d}x \neq 0. \tag{3.275}$$

Proof. The case $l = 0$ is clear. Let $l \in \mathbb{N}$. Then it follows from (2.322) with $k = 0$ that

$$h^{l,1}(x) = \int_{-\infty}^{x} h^l(y)\,\mathrm{d}y = -\int_{x}^{\infty} h^l(y)\,\mathrm{d}y, \quad x \in \mathbb{R}, \tag{3.276}$$

and

$$\left|h^{l,1}(x)\right| + \left| \frac{\mathrm{d}}{\mathrm{d}x} h^{l,1}(x) \right| \le c\, e^{-\alpha |x|}, \quad x \in \mathbb{R}. \tag{3.277}$$

Let $k = 0, \dots, l - 1$. Then

$$\begin{aligned} x^{k+1} h^{l,1}(x) &= \int_{-\infty}^{x} \big(y^{k+1} h^{l,1}(y)\big)'\,\mathrm{d}y \\ &= \int_{-\infty}^{x} (k+1) y^k h^{l,1}(y)\,\mathrm{d}y - \int_{x}^{\infty} y^{k+1} h^l(y)\,\mathrm{d}y, \end{aligned} \tag{3.278}$$

where we used (2.322). Now $x \to \infty$ shows that

$$\int_{-\infty}^{\infty} y^k h^{l,1}(y)\,\mathrm{d}y = 0 \quad \text{where} \quad k = 0, \dots, l - 1. \tag{3.279}$$

Iteration

$$h^{l,r}(x) = \int_{-\infty}^{x} h^{l,r-1}(y)\,\mathrm{d}y, \quad v^l(x) = 2\int_{-\infty}^{x} h^{l,l}(y)\,\mathrm{d}y, \quad x \in \mathbb{R}, \tag{3.280}$$

with $r = 1, \ldots, l$ and the properties of $h^l = h^l_M$ according to Section 2.5.1 prove all assertions for v^l with the exception of (3.275). But this follows from Theorem 3.40 below by contradiction. □

Definition 3.38. Let $l \in \mathbb{N}_0$ and let v^l be a basic Faber spline of order l according to (3.273). Then

$$\{v^l_{jm}(x) = v^l\big(2^j x - m\big) : j \in \mathbb{N}_0,\ m \in \mathbb{Z}\} \tag{3.281}$$

is called a *Faber spline system* of order l.

Remark 3.39. Let v_{jm} be the same functions as in (3.3). Then it follows from (3.272) that

$$\{v_{jm} : j \in \mathbb{N}_0,\ m \in \mathbb{Z}\} \tag{3.282}$$

is a Faber spline system of order 0. It is the classical Faber system (3.1) without the functions v_0 and v_1, but extended from I to $\mathbb{R}$.

Let $I^* = (0, 1/2)$ be half of the unit interval $I = (0,1)$ on $\mathbb{R}$. Let $\widetilde{B}^s_{pq}(I^*)$ be the spaces introduced in Definition 1.24 (ii). We are interested only in spaces with $s > 1/p$. Then it follows from (1.92), (1.93) that

$$\widetilde{B}^s_{pq}(I^*) = \big\{ f \in B^s_{pq}(\mathbb{R}) : \operatorname{supp} f \subset \overline{I^*} \big\} \tag{3.283}$$

makes sense. Now we extend the second preparation in Section 3.5.1 from Haar system, Faber system (of order 0) to spline systems, Faber systems of order $l \in \mathbb{N}$. As said above we are more interested in the description of key ideas than in technical details. Otherwise we extend some assertions of Theorem 3.1. Recall that $s > 1/p$ ensures (3.10), hence the continuous embedding of $B^s_{pq}(I^*)$ and $\widetilde{B}^s_{pq}(I^*)$ into $C(I^*)$.

Theorem 3.40. *Let $l \in \mathbb{N}_0$ and let $\{v^l_{jm}\}$ be the Faber spline system of order l according to Definition 3.38. Let $0 < p, q \le \infty$ and*

$$\frac{1}{p} < s < 2l + 1 + \min\Big(\frac{1}{p}, 1\Big), \tag{3.284}$$

Figure 3.3, p. 170. Then $f \in \widetilde{B}^s_{pq}(I^*)$ can be represented as

$$f(x) = \sum_{j \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}} \sum_{k=1}^{2^j - 1} a^l_{km}\, f\big(2^{-j-1}k\big)\, v^l_{jm}(x), \quad x \in I^*, \tag{3.285}$$

unconditional convergence being in $C(I^*)$, where $a^l_{km} \in \mathbb{R}$ with

$$|a^l_{km}| \le c\, e^{-\beta |2m-k|}, \quad m \in \mathbb{Z};\ k \in \mathbb{N}, \tag{3.286}$$

for some $\beta > 0$ and $c > 0$.

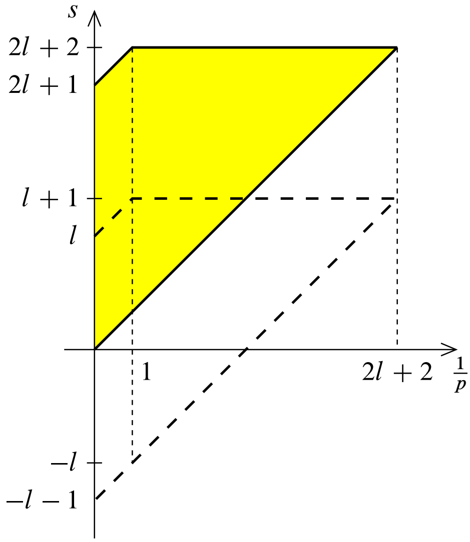

Figure 3.3. Faber splines.

Proof. Using (3.283) it follows that

$$f^{(l+1)}(x) \in B^{\sigma}_{pq}(\mathbb{R}), \quad \operatorname{supp} f^{(l+1)} \subset \overline{I^*} \tag{3.287}$$

with

$$0 < p, q \leq \infty, \qquad \frac{1}{p} - 1 - l < \sigma < l + \min\Big(\frac{1}{p}, 1\Big). \tag{3.288}$$

One can apply Theorem 2.49 (i) and Theorem 2.46 (i) with σ in place of s. Then

$$f^{(l+1)}(x) = \sum_{j=-1}^{\infty} \sum_{m \in \mathbb{Z}} 2^j \, h^l_{jm}(x) \int_{\mathbb{R}} f^{(l+1)}(y) \, h^l_{jm}(y) \, \mathrm{d}y. \tag{3.289}$$

Let $j \in \mathbb{N}_0$. Then one has by (2.323) and (3.273) that

$$\frac{\mathrm{d}^l}{\mathrm{d}x^l} h^l_{jm}(x) = 2^{jl} \frac{\mathrm{d}h^l}{\mathrm{d}x^l}(2^j x - m) = 2^{jl-1} \frac{\mathrm{d}^{2l+1} v^l}{\mathrm{d}x^{2l+1}}(2^j x - m). \tag{3.290}$$

These functions are constant on the intervals $\big(2^{-j-1}k, 2^{-j-1}(k+1)\big)$. Using integration by parts it follows (at least for smooth functions) that

$$2^j \int_{\mathbb{R}} f^{(l+1)}(y) \, h^l_{jm}(y) \, \mathrm{d}y = (-1)^l \, 2^{j(l+1)-1} \sum_{k \in \mathbb{Z}} b^l_{km} f\big(2^{-j-1}k\big) \tag{3.291}$$

with

$$\begin{aligned} b^l_{km} &= (v^l)^{(2l+1)}\big(2^j(2^{-j-1}k-2^{-j-1})-m\big)-(v^l)^{(2l+1)}\big(2^j2^{-j-1}k-m\big) \\ &= (v^l)^{(2l+1)}\big(\tfrac{k}{2}-\tfrac12-m\big)-(v^l)^{(2l+1)}\big(\tfrac{k}{2}-m\big). \end{aligned} \tag{3.292}$$

If $j=-1$ then one has $f(k/2)$ in the counterpart of (3.291). But this is zero by assumption. Hence (3.289) reduces to the terms with $j\in\mathbb{N}_0$ (even with $j\in\mathbb{N}$). This can be extended to arbitrary functions f in (3.287) by approximation (in Section 3.5.3 below we add a few comments about convergence). One has by (3.281) and (3.273),

$$h^l_{jm}(x)=h^l(2^jx-m)=2^{-1}\Big(\frac{\mathrm{d}^{l+1}}{\mathrm{d}x^{l+1}}v^l\Big)(2^jx-m)=2^{-1-j(l+1)}\big(v^l_{jm}\big)^{(l+1)}(x). \tag{3.293}$$

We insert (3.291), (3.293) in (3.289). Using Proposition 3.37 one obtains by integration that

$$f(x)=(-1)^l2^{-2}\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}b^l_{km}\,f\big(2^{-j-1}k\big)\,v^l_{jm}(x). \tag{3.294}$$

This proves (3.285) where (3.286) follows from (3.292) and (3.274). For the expansion of $f^{(l+1)}$ in (3.289) one has the convergence properties according to Theorems 2.46, 2.49. This can be transferred to corresponding convergence assertions for f in (3.285). (We add a few comments about convergence in Section 3.5.3 below.) In particular, this series converges unconditionally in $C(I^*)$. □

Remark 3.41. If $l=0$ then $\{v^0_{jm}\}=\{v_{jm}\}$ is the classical Faber system (3.282) = (3.3). In this case one can replace $I^*=(0,1/2)$ in the above theorem by $I=(0,1)$. But this is unimportant. According to Theorem 3.1 any

$$f\in\widetilde{B}^s_{pq}(I^*),\quad 0<p,q\le\infty,\quad \tfrac1p<s<1+\min\big(\tfrac1p,1\big), \tag{3.295}$$

can be represented by

$$f(x)=-\frac12\sum_{j\in\mathbb{N}_0}\sum_{m=0}^{2^j-1}\big(\Delta^2_{2^{-j-1}}f\big)(2^{-j}m)\,v_{jm}(x) \tag{3.296}$$

with the indicated convergence properties. In particular

$$\sum_{k=1}^{2^j-1}a^0_{km}\,f(2^{-j-1}k)=-\tfrac12\big(\Delta^2_{2^{-j-1}}f\big)(2^{-j}m),\quad j\in\mathbb{N}_0,\ m\in\mathbb{Z}, \tag{3.297}$$

where only $m=0,\ldots,2^j-1$ is relevant. The function v_{jm} is supported near $2^{-j}m$ and the related function values of f needed are also in a neighbourhood of $2^{-j}m$. If $l\in\mathbb{N}$ then the situation is different. By (3.281) and Proposition 3.37 the functions v^l_{jm} are concentrated near $2^{-j}m$ from where they decay exponentially, but they do not have compact supports. According to (3.286) the numbers a^l_{km} concentrate at $k = 2m$. This shows that the factor of v^l_{jm} in (3.285),

$$\sum_{k=1}^{2^j-1} a^l_{km}\, f\big(2^{-j-1}k\big), \quad \text{concentrates at } 2^{-j}m, \tag{3.298}$$

and decays exponentially from there. This is a reasonable substitute of the corresponding compactness assertions in (3.297).

3.5.3 Comments, problems, proposals

The above Theorem 3.40 has not the same final character as Theorem 3.1 in case of the Faber system $\{v_{jm}\}$ even if f is restricted to (3.296) (which means $f(0) = f(1) = 0$). It was our main aim to make clear that there are expansions of type (3.285) in spaces with higher smoothness based on pointwise evaluation. One can expect that this observation paves the way to extend the theory developed in this Chapter 3 for the Faber system $\{v_{jm}\}$ to the Faber spline systems $\{v^l_{jm}\}$ of any order $l \in \mathbb{N}_0$. Nothing has been done so far and these remarks may also be considered as a proposal for future research. In this sense we add a few comments.

1. Convergence. First we comment on the representation of $f \in \widetilde{B}^s_{pq}(I^*)$ with p, q, s as in Theorem 3.40. Let b_{pq} be the sequence spaces according to (2.41), (2.42). We suppose that f can be expanded in $S'(\mathbb{R})$ by the unconditionally convergent series

$$f = \sum_{j\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}} \mu_{jm}\, 2^{-j(s-\frac{1}{p})} v^l_{jm}. \tag{3.299}$$

Then $f^{(l+1)} \in B^{s-l-1}_{pq}(\mathbb{R})$ can be represented in $S'(\mathbb{R})$ by the unconditionally convergent series

$$f^{(l+1)} = 2 \sum_{j\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}} \mu_{jm}\, 2^{-j(s-l-1-\frac{1}{p})}\, h^l_{jm} \tag{3.300}$$

where we used (3.281), (3.273) and (2.323). By (3.287) and (3.288) with $\sigma = s-l-1$ it follows from Theorem 2.49 that

$$\mu = \big\{\mu_{jm} : j \in \mathbb{N}_0;\ m \in \mathbb{Z}\big\} \in b_{pq}. \tag{3.301}$$

This representation is unique. Compared with (3.285) one obtains that

$$\mu_{jm} = 2^{j(s-\frac{1}{p})} \sum_{k=1}^{2^j-1} a^l_{km} f\big(2^{-j-1}k\big) = 2^{j(s-\frac{1}{p})} D^l_{jm} f \tag{3.302}$$

indicating notationally that these coefficients are finite distinguished linear combinations of the function values $f(2^{-j-1}k)$. If $l = 0$ then we have (3.297). Furthermore,

$$\Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{1}{p})q} \Big(\sum_{m\in\mathbb{Z}} |D^l_{jm} f|^p\Big)^{q/p}\Big)^{1/q} \sim \|f \,|B^s_{pq}(\mathbb{R})\| \tag{3.303}$$

where the equivalence constants are independent of $f \in \widetilde{B}^s_{pq}(I^*)$, quasi-normed according to (3.283) (usual modification if $p = \infty$ and/or $q = \infty$). This extends essentially the discrete quasi-norm in the last line of (3.42) from $l = 0$ to $l \in \mathbb{N}_0$ (under the above specific situation which means $f(0) = f(1) = 0$). Formally one can derive the above assertions first for functions belonging to $D(I^*) = C_0^\infty(I^*)$. This set is dense in $\widetilde{B}^s_{pq}(I^*)$ if $p < \infty, q < \infty$. The rest is a matter of completion. Afterwards one can incorporate the cases with $\max(p, q) = \infty$.

2. Full spaces. It would be desirable to extend the above considerations to $B^s_{pq}(I^*)$ (and $F^s_{pq}(I^*)$ with the Sobolev spaces $H^s_p(I^*)$, $1 < p < \infty$, as special cases). We outlined in Preparation 1 in Section 3.5.1 a way how to reduce the full spaces $B^s_{pq}(I^*)$ to spaces as considered in Theorem 3.40. This is somewhat in contrast to Theorem 3.1 where we dealt directly with Faber bases in $B^s_{pq}(I)$. But if s is large, say, $s > 1 + \frac{1}{p}$, then one must find a way how to deal with, say, $f'(0)$ (and maybe higher derivatives). This does not fit immediately in the above context. Preparation 1 in Section 3.5.1 circumvents this obstacle.

3. Higher dimensions. In Section 3.2 we dealt with Faber bases for spaces with dominating mixed smoothness in cubes. We refer in particular to Theorems 3.13, 3.16 and Section 3.2.5. Our approach relied on a corresponding theory for expansions in terms of Haar tensor bases as presented in Section 2.4. We described in Section 2.5.4 how this theory can be extended to spline tensor bases in higher dimensions. This may serve as a fundamental to develop a corresponding theory for Faber splines in suitable spaces with dominating mixed smoothness in higher dimensions.

4. Sampling and integration. In the following chapters of this book we deal with sampling, numerical integration and discrepancy, especially on intervals and cubes. This will be based on Faber expansions with Theorems 3.1, 3.13, 3.16 as the main references. This causes the restrictions

$$0 < p, q \le \infty, \quad \frac{1}{p} < s < 1 + \min\left(\frac{1}{p}, 1\right), \tag{3.304}$$

($r = s$ in higher dimensions) for the source spaces. If one replaces Faber systems by Faber spline systems of order $l \in \mathbb{N}_0$ according to Definition 3.38 then there is a reasonable chance to extend this theory from spaces with p, q, s as in (3.304) to corresponding source spaces with

$$0 < p, q \le \infty, \quad \frac{1}{p} < s < 2l + 1 + \min\left(\frac{1}{p}, 1\right), \quad l \in \mathbb{N}_0, \tag{3.305}$$

Figure 3.3, p. 170.

Chapter 4
Sampling

4.1 Definitions, sampling in isotropic spaces

4.1.1 Definitions

This Chapter 4 deals with sampling numbers as an application of expansions of functions by Faber bases, treated in the preceding Chapter 3. First we collect some definitions.

As usual $T\colon A \hookrightarrow B$ means that T is a linear and bounded (continuous) operator (map) from the quasi-Banach space A into the quasi-Banach space B. If $A \subset B$ then $\mathrm{id}\colon A \hookrightarrow B$ stands for the identity (embedding) $a = \mathrm{id}\, a \in B$ with $a \in A$, also abbreviated by $A \hookrightarrow B$.

Let Ω be a bounded domain in $\mathbb{R}^n$. Recall that *domain* means open set. We are interested in compact embeddings

$$\mathrm{id}\colon G_1(\Omega) \hookrightarrow G_2(\Omega), \tag{4.1}$$

where $\{G_1(\Omega), G_2(\Omega)\} \subset D'(\Omega)$ are either distributional quasi-Banach spaces or $\{G_1(\Omega), G_2(\Omega)\} \subset \mathbf{M}(\Omega)$ are quasi-Banach spaces of Lebesgue-measurable functions. As usual, $D'(\Omega)$ is the space of all complex-valued distributions on Ω, whereas $\mathbf{M}(\Omega)$ is the collection of all equivalence classes of Lebesgue almost everywhere finite complex-valued functions in Ω, furnished with the *convergence in measure* and converted into a complete metric space. We refer to Section 1.1.8 where we gave a more detailed description with a reference to [Mall95, Section I,5]. Recall that $C(\Omega)$ is the collection of all complex-valued continuous functions in $\overline{\Omega}$, furnished in the usual way with a norm, Definition 1.24 (iii), Remark 1.25. We always assume that the *source space* $G_1(\Omega)$ is continuously embedded in $C(\Omega)$,

$$\mathrm{id}\colon G_1(\Omega) \hookrightarrow C(\Omega), \tag{4.2}$$

interpreted as usual: In each equivalence class of $G_1(\Omega)$ there is a (uniquely determined) continuous function in $\overline{\Omega}$ to which (4.2) applies.

Next we describe the basic ingredients of *sampling methods* for the compact embedding (4.1), (4.2). Let $\{x^j\}_{j=1}^k \subset \Omega$. By (4.2) the *information map*

$$N_k\colon G_1(\Omega) \mapsto \mathbb{C}^k, \quad k \in \mathbb{N}, \tag{4.3}$$

given by

$$N_k f = \big(f(x^1), \dots, f(x^k)\big), \quad f \in G_1(\Omega), \tag{4.4}$$

according to the above interpretation makes sense. As usual, $\mathbb{C}^k$ is the collection of all k-tuples of complex numbers. Let

$$S_k = \Phi_k \circ N_k, \quad \text{where } \Phi_k : \mathbb{C}^k \mapsto G_2(\Omega) \tag{4.5}$$

be an arbitrary map (also called *method* or *algorithm*). Hence

$$S_k f = \Phi_k\big(f(x^1), \dots, f(x^k)\big) \in G_2(\Omega), \quad f \in G_1(\Omega). \tag{4.6}$$

One wishes to recover a given continuous function $f \in G_1(\Omega)$ in $G_2(\Omega)$ by asking for optimally scattered points $\{x^j\}_{j=1}^k$ and optimally chosen methods Φ_k.

Definition 4.1. Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $G_1(\Omega)$, $G_2(\Omega)$ be quasi-Banach spaces either of distributional spaces, $\{G_1(\Omega), G_2(\Omega)\} \subset D'(\Omega)$, or of Lebesgue-measurable spaces, $\{G_1(\Omega), G_2(\Omega)\} \subset \mathbf{M}(\Omega)$, satisfying (4.1), (4.2). Let $k \in \mathbb{N}$.

(i) Then

$$g_k(\mathrm{id}) = \inf \Big[\sup \big\{ \|f - S_k f \,|G_2(\Omega)\| : \|f \,|G_1(\Omega)\| \le 1 \big\} \Big] \tag{4.7}$$

is the k-th *sampling number*, where the infimum is taken over all k-tuples $\{x^j\}_{j=1}^k \subset \Omega$ and all maps $S_k = \Phi_k \circ N_k$ according to (4.3)–(4.6).

(ii) The *linear sampling numbers* $g_k^{\mathrm{lin}}(\mathrm{id})$ are given by (4.7) where the infimum is taken over all k-tuples $\{x^j\}_{j=1}^k \subset \Omega$ and all linear maps $S_k = \Phi_k \circ N_k$ with

$$S_k f = \sum_{j=1}^k f(x^j)\, h_j, \quad h_j \in G_2(\Omega), \; f \in G_1(\Omega). \tag{4.8}$$

Remark 4.2. This is an adapted version of a corresponding definition in [NoT06], repeated in [T06, pp. 218/219]. Obviously,

$$g_k(\mathrm{id}) \le g_k^{\mathrm{lin}}(\mathrm{id}), \quad k \in \mathbb{N}. \tag{4.9}$$

There are many other numbers to quantify the compactness of (4.1). But this is not subject of this book. As indicated above we are almost exclusively interested in an application of Faber bases to sampling numbers. A detailed discussion of several types of quantifications of (4.1) from the point of view of *information-based complexity* may be found in [NoW08, Chapter 4] and [NoW09]. The pointwise evaluation of f as in (4.4)–(4.6) is called there the *standard information* and indicated by Λ^{std}. Asking for numbers of type (4.7) admitting all $f \in G_1(\Omega)$ in the unit ball of $G_1(\Omega)$ is called the *worst case setting*. In the notation used in [NoW08], [NoW09] one has

$$e^{\mathrm{wor}}\big(k, \Lambda^{\mathrm{std}}\big) = g_k(\mathrm{id}) \quad \text{and} \quad e^{\mathrm{wor\text{-}lin}}\big(k, \Lambda^{\mathrm{std}}\big) = g_k^{\mathrm{lin}}(\mathrm{id}), \tag{4.10}$$

$k \in \mathbb{N}$. We refer to [NoW08, pp. 93/94, 117].

Let I be an (arbitrary) index set. Then

$$a_i \sim b_i \quad \text{for } i \in I \quad \text{(equivalence)} \tag{4.11}$$

for two sets of positive numbers $\{a_i : i \in I\}$ and $\{b_i : i \in I\}$ means that there are two positive numbers c_1 and c_2 such that

$$c_1\, a_i \le b_i \le c_2\, a_i \quad \text{for all } i \in I. \tag{4.12}$$

Let id be the compact embedding (4.1) and let $\Gamma = \{x^j\}_{j=1}^l \subset \Omega$. Then

$$\mathrm{id}^\Gamma : G_1^\Gamma(\Omega) = \Big\{ f \in G_1(\Omega) : \sum_{j=1}^l |f(x^j)| = 0 \Big\} \hookrightarrow G_2(\Omega) \tag{4.13}$$

is a compact operator and its quasi-norm $\|\mathrm{id}^\Gamma\|$ has the usual meaning.

Proposition 4.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let $G_1(\Omega)$, $G_2(\Omega)$ and* id *be as in Definition 4.1. Let* id^Γ *be given by (4.13). Then*

$$g_k(\mathrm{id}) \sim \inf \Big\{ \|\mathrm{id}^\Gamma\| : \operatorname{card} \Gamma \le k \Big\}, \quad k \in \mathbb{N}, \tag{4.14}$$

where the equivalence constants are independent of k.

Remark 4.4. This coincides essentially with [NoT06, Proposition 19] and [T06, Proposition 4.34, p. 220]. In case of Banach spaces we refer also to [TWW88, pp. 45, 58].

Plan of this chapter. In Sections 4.2, 4.3, 4.4 we deal with sampling numbers for spaces on intervals, spaces with dominating mixed smoothness, and logarithmic spaces, respectively, always assuming that the corresponding source spaces $G_1(\Omega)$ admit expansions in terms of Faber bases according to the related sections in Chapter 3. This restricts the integrability parameter p and the smoothness parameters s, r in the source spaces $G_1(\Omega)$ of type A^s_{pq} and $S^r_{pq}B$ in a decisive way. In other words, we are more interested in applications of Faber bases than in a systematic study of sampling numbers in related spaces. But the results obtained will be of some use for the later considerations about numerical integration and discrepancy. However we complement this introductory Section 4.1 by a report on some recent results for sampling numbers in isotropic spaces, Section 4.1.2, and by a discussion of unavoidable *information uncertainties* how large the target spaces $G_2(\Omega)$ may be chosen (in sharp contrast to all other numbers and widths known to us), Section 4.1.3.

4.1.2 Sampling in isotropic spaces

We describe some recent developments of the theory of sampling numbers in isotropic spaces. No proofs are given, but comments and references.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$, according to Definition 1.26 (ii) or a bounded interval in $\mathbb{R}$. Let $A^s_{pq}(\Omega)$ be the isotropic spaces as introduced in

Definition 1.24 (i) by restriction of $A^s_{pq}(\mathbb{R}^n)$ to Ω. Recall that $C(\Omega)$ is the collection of all complex-valued continuous functions on $\overline{\Omega}$, furnished with the norm (1.94). The n-dimensional counterpart of (3.8), (3.9) is given by

$$B^s_{pq}(\Omega) \hookrightarrow C(\Omega) \quad \text{if, and only if,} \quad \begin{cases} 0 < p \le \infty,\ 0 < q \le \infty, & s > n/p, \\ 0 < p \le \infty,\ 0 < q \le 1, & s = n/p, \end{cases} \tag{4.15}$$

and

$$F^s_{pq}(\Omega) \hookrightarrow C(\Omega) \quad \text{if, and only if,} \quad \begin{cases} 0 < p < \infty,\ 0 < q \le \infty, & s > n/p, \\ 0 < p \le 1,\ 0 < q \le \infty, & s = n/p. \end{cases} \tag{4.16}$$

We refer to [T08, Section 6.4, p. 229]. But the assertion itself is well known. One may consult [T01, Section 11] and [Har07, Section7.2]. The limiting cases go back to [SiT95]. One can specify $G_1(\Omega)$ in (4.1), (4.2) by any of the spaces in (4.15), (4.16). Let

$$0 < p, q \le \infty, \quad s > n/p \quad \text{and} \quad 0 < r \le \infty. \tag{4.17}$$

Then

$$\mathrm{id}\colon B^s_{pq}(\Omega) \hookrightarrow L_r(\Omega), \tag{4.18}$$

considered as subspaces of $\mathbf{M}(\Omega)$, is compact, and

$$g_k(\mathrm{id}) \sim g^{\mathrm{lin}}_k(\mathrm{id}) \sim k^{-\frac{s}{n}+(\frac{1}{p}-\frac{1}{r})_+}, \quad k \in \mathbb{N}. \tag{4.19}$$

Here $a_+ = \max(a, 0)$ for $a \in \mathbb{R}$. This is one of the main assertions in [NoT06] and may also be found in [T06, Theorem 4.37, p. 224].

Problems of sampling and optimal recovery have some history where numbers of type (4.10) and numerous modifications play a decisive role. As far as the general context is concerned we refer to [NoW08], [NoW09]. Sampling problems have been studied by many authors, preferably for functions on cubes $\Omega = \mathbb{Q}^n = (0,1)^n$ or periodic functions on the n-torus $\mathbb{T}^n$ and with $C^k(\Omega)$, Hölder–Zygmund spaces $\mathcal{C}^s(\Omega)$, classical Sobolev spaces $W^k_p(\Omega)$ or classical Besov spaces as source spaces $G_1(\Omega)$. We refer to [Cia78], [Nov88], [Hei94], [Kud93], [Kud95], [Kud98], [Tem93], [BNR99]. In case of bounded Lipschitz domains we refer to [Wen01], [NWW05]. A survey of some recent results, also for other numbers than in (4.10), has been given in [NoW08, Section 4.2.4, pp. 119–127].

Let again Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$, or a bounded interval on $\mathbb{R}$ ($n = 1$). Assertions of type (4.17)–(4.19) can be extended to other admitted couples $\{G_1(\Omega), G_2(\Omega)\}$. Let

$$0 < p_1, q_1 \le \infty, \quad s > n/p_1, \tag{4.20}$$

and

$$0 < p_2, q_2 \le \infty, \quad n\Big(\frac{1}{p_2} - 1\Big)_+ < s_2 < s_1 - n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+ \tag{4.21}$$

(with $p_1 < \infty$, $p_2 < \infty$ in case of F-spaces). Then

$$\mathrm{id}\colon A^{s_1}_{p_1 q_1}(\Omega) \hookrightarrow A^{s_2}_{p_2 q_2}(\Omega), \tag{4.22}$$

considered as subspaces of $D'(\Omega)$, is compact, and

$$g_k(\mathrm{id}) \sim k^{-\frac{s_1-s_2}{n}+(\frac{1}{p_1}-\frac{1}{p_2})_+}, \quad k \in \mathbb{N}, \tag{4.23}$$

Figure 4.1. We refer to [Tri05] and [T06, Theorem 4.40, p. 226]. It remained open whether (4.23) can be complemented by $g_k^{\mathrm{lin}}(\mathrm{id}) \sim g_k(\mathrm{id})$ as the expected counterpart

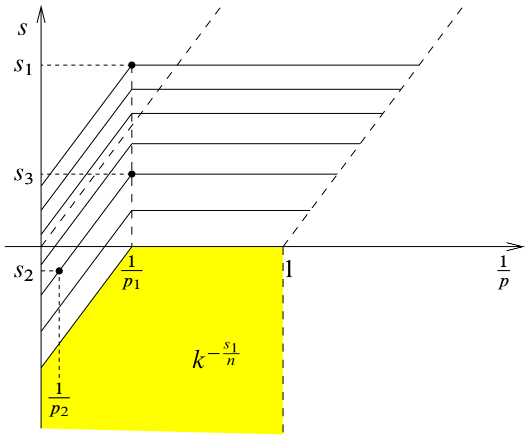

Figure 4.1. Sampling numbers g_k.

of (4.19). Restricted to (fractional) Sobolev spaces $H^s_p(\Omega) = F^s_{p,2}(\Omega)$, $1 < p < \infty$, $s > 0$, the related equivalence was stated in [NoW08, Open Problem 18, p. 123] as a conjecture. Affirmative answers extending (4.23) to

$$g_k(\mathrm{id}) \sim g_k^{\mathrm{lin}}(\mathrm{id}) \sim k^{-\frac{s_1-s_2}{n}+(\frac{1}{p_1}-\frac{1}{p_2})_+}, \quad k \in \mathbb{N}, \tag{4.24}$$

are due to J. Vybíral, [Vyb07], ($\Omega = \mathbb{Q}^n$ cube in $\mathbb{R}^n$, $1 \le p_1, p_2, q_1, q_2$) and S. Heinrich, [Hei08a], [Hei08b, Theorem 5.3] (Ω bounded Lipschitz domain, Besov and Sobolev spaces, $1 \le p_1, p_2, q_1, q_2$). The other cases remained open. This applies also to (4.23) and even more to (4.24) for other admitted target spaces $A^{s_2}_{p_2 q_2}(\Omega)$ not covered by (4.21). But at least a few results have been obtained recently which we mention now briefly.

According to (4.15), (4.16) the assumption (4.20) ensures (4.2). What about limiting cases with $s = n/p$ covered by (4.15), (4.16)? This has been discussed in [T08, Theorem 6.79, Remark 6.80, pp. 235–236] (with some restrictions). It comes out that assertions of type (4.23) remain valid for these limiting cases. So far we always assumed that $s_2 > n\big(\frac{1}{p_2} - 1\big)_+$ for the target spaces. What happens if $s_2 = n\big(\frac{1}{p_2} - 1\big)_+$, in particular $s_2 = 0$, $1 \le p_2 \le \infty$? This case has been studied in [Vyb07] and [Vyb08a, Theorem 4.2]. It comes out that the expected behaviour $k^{-s/n}$ both of $g_k(\mathrm{id})$ and $g_k^{\mathrm{lin}}(\mathrm{id})$ for

$$\mathrm{id}\colon B^s_{pq_1}(\mathbb{Q}^n) \hookrightarrow B^0_{pq_2}(\mathbb{Q}^n), \quad s > n/p, \ 1 \le p, q_1, q_2 \le \infty \tag{4.25}$$

is perturbed by factors $(\log k)^\lambda$, $k \ge 2$.

Let Ω be again a bounded Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$ (or a bounded interval in $\mathbb{R}$). Then

$$\mathrm{id}\colon A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow A^{s_2}_{p_2q_2}(\Omega) \tag{4.26}$$

is compact if, and only if,

$$0 < p_1, p_2 \le \infty, \quad -\infty < s_2 < s_1 - n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+, \tag{4.27}$$

$0 < q_1, q_2 \le \infty$ (with $p_1 < \infty$ and $p_2 < \infty$ for F-spaces). The degree of compactness can be expressed in terms of entropy numbers or diverse s-numbers (including approximation numbers, Kolmogorov numbers, Gelfand numbers). The respective expressions depend for fixed p_1, p_2, q_1 , q_2 only on the difference $s_1 - s_2$, but not on the absolute values of s_1 and s_2. The situation for sampling numbers is different. On the one hand one must ensure (4.2). This means (4.20) or at least (4.15), (4.16) if the source space $G_1(\Omega) = A^{s_1}_{p_1q_1}(\Omega)$ is an isotropic space. Then one has (4.23) or (at the best) (4.24) but under the restriction $s_2 > n\big(\frac{1}{p_2} - 1\big)_+$ for the target spaces. What about $s_2 < n\big(\frac{1}{p_2} - 1\big)_+$, especially $s_2 < 0$ if $1 \le p_2 \le \infty$? It comes out that there is no improvement if one admits target spaces $G_2(\Omega) = A^{s_2}_{p_2q_2}(\Omega)$ with $s_2 < 0$ and that nothing like (4.23) or even (4.24) can be expected, in sharp contrast to the above-mentioned numbers (entropy, approximation, Kolmogorov, Gelfand). Even worse, there is an unavoidable *information uncertainty* which cannot be improved how negative s_2 might be chosen. We return to this question in detail in the next Section 4.1.3. At this moment we restrict ourselves to a discussion what can be said if one steps from $s_2 > 0$ to $s_2 < 0$ using well-known embedding theorems.

Let again Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$ (or a bounded interval in $\mathbb{R}$) and let

$$\mathrm{id}\colon A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow A^{s_2}_{p_2q_2}(\Omega), \quad p_1, p_2 \ge 1, \ s_1 > n/p_1, \ s_2 < 0. \tag{4.28}$$

If, in addition, $-\frac{n}{p_1} > s_2 - \frac{n}{p_2}$ then it follows from

$$\mathrm{id}\colon A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow L_{p_1}(\Omega) \hookrightarrow A^{s_2}_{p_2q_2}(\Omega) \tag{4.29}$$

and (4.19) that

$$g_k(\mathrm{id}) \le g_k^{\mathrm{lin}}(\mathrm{id}) \le c\, k^{-s_1/n}, \quad k \in \mathbb{N}. \tag{4.30}$$

If, in addition, $-\frac{n}{p_1} < s_2 - \frac{n}{p_2}$, Figure 4.1, p. 178, then one has for $s_3 = s_2 - \frac{n}{p_2} + \frac{n}{p_1} > 0$ that

$$\mathrm{id}\colon A^{s_1}_{p_1q_1}(\Omega) \hookrightarrow A^{s_3}_{p_1q_2}(\Omega) \hookrightarrow A^{s_2}_{p_2q_2}(\Omega) \tag{4.31}$$

and by (4.9) and (4.24) (at least for cubes)

$$g_k(\mathrm{id}) \le g_k^{\mathrm{lin}}(\mathrm{id}) \le c\, k^{-\frac{s_1-s_3}{n}} = c\, k^{-\frac{s_1-s_2}{n}+\frac{1}{p_1}-\frac{1}{p_2}}, \quad k \in \mathbb{N}. \tag{4.32}$$

It has been observed by J. Vybiral in [Vyb07, Theorem 2.8] that these assertions are sharp. In other words, one has for id in (4.28) that

$$g_k(\mathrm{id}) \sim g_k^{\mathrm{lin}}(\mathrm{id}) \sim k^{-s_1/n} \quad \text{if } -\frac{n}{p_1} > s_2 - \frac{n}{p_2}, \quad k \in \mathbb{N}, \tag{4.33}$$

and

$$g_k(\mathrm{id}) \sim g_k^{\mathrm{lin}}(\mathrm{id}) \sim k^{-\frac{s_1-s_2}{n}+\frac{1}{p_1}-\frac{1}{p_2}} \quad \text{if } -\frac{n}{p_1} < s_2 - \frac{n}{p_2}, \quad k \in \mathbb{N}, \tag{4.34}$$

(at least for cubes), Figure 4.1, p. 178. We refer in this context also to the recent paper [Hei09, Theorem 5.1 and Section 6].

We summarise the outcome. Let again Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$, or a bounded interval in $\mathbb{R}$. Let

$$\mathrm{id}\colon A^{s_1}_{p_1 q_1}(\Omega) \hookrightarrow A^{s_2}_{p_2 q_2}(\Omega), \quad p_1 \ge 1,\ s_1 > n/p_1,\ 0 < q_1 \le \infty, \tag{4.35}$$

and

$$s_1 - s_2 > n\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+, \quad s_2 \in \mathbb{R},\ 0 < p_2, q_2 \le \infty, \tag{4.36}$$

($p_1, p_2 < \infty$ in case of F-spaces). Then one has for $g_k(\mathrm{id})$ the level lines shown in Figure 4.1, p. 178. The same for $g_k^{\mathrm{lin}}(\mathrm{id})$ (at least for cubes in case of general spaces). *There are some white spots*. We ***conjecture*** that the horizontal level lines for $s_2 > 0$ and $p_2 < p_1$ can be extended to all $p_2 < p_1$ both for $g_k(\mathrm{id})$ and $g_k^{\mathrm{lin}}(\mathrm{id})$. This will be confirmed in Section 4.2.2 in some special cases. Similarly one may ask whether $g_k(\mathrm{id}) \sim g_k^{\mathrm{lin}}(\mathrm{id}) \sim k^{-s_1/n}$ is also valid for the remaining regions with $s_2 < 0$. But this is the case. It follows from the considerations in the next Section 4.1.3. We refer to Remark 4.9. If $p_1 < 1$ then the situation is even worse. One has (4.18) with $p_1 = r < 1$ in the framework of $\mathbf{M}(\Omega)$. But one cannot argue as in (4.29) what requires $D'(\Omega)$ in place of $\mathbf{M}(\Omega)$ as the related framework. In other word, *there are even more white spots*. Despite some gaps the above considerations suggest the following conclusion.

> *In sharp contrast to entropy numbers and s-numbers (including approximation numbers, Kolmogorov numbers and Gelfand numbers) the sensitive regions of the source spaces $A^{s_1}_{p_1 q_1}(\Omega)$ and of the target spaces $A^{s_2}_{p_2 q_2}(\Omega)$ for sampling numbers $g_k(\mathrm{id})$ and $g_k^{\mathrm{lin}}(\mathrm{id})$ in bounded Lipschitz domains Ω are $s_1 > \frac{n}{p_1}$ and $s_2 > n\big(\frac{1}{p_2} - 1\big)_+$, respectively. The rest is a matter of embeddings and interesting but sophisticated limiting and border-line cases.*

Bounded Lipschitz domains seem to be the natural choice in the context of sampling numbers (beyond cubes). But it is not necessary. For some types of spaces it is sufficient to assume that Ω is a (maybe unbounded) so-called E-thick domain or even an arbitrary domain with $|\Omega| < \infty$. We refer to [Tri07] and Section 6.5.3, pp. 240–242, in [T08]. We compared sampling numbers with other numbers. Here are a few relevant references where one finds further information and also historical comments about other numbers. Final assertion for entropy numbers in arbitrary bounded domains may be found in [T06, Theorem 1.97, Remark 1.98, p. 61]. As far as s-numbers are concerned (especially approximation numbers, Kolmogorov numbers, Gelfand numbers) we refer to the survey [Vyb08b]. As for the relevance of these assertions in the context of numerical analysis and complexity theory one may consult [NoW08], where one finds in [NoW08, Section 4.2.4, pp. 119–127] a description of corresponding results restricted to (fractional) Sobolev spaces.

4.1.3 Information uncertainty

Let the source space $A^{s_1}_{p_1q_1}(\Omega)$ in (4.35), (4.36) be fixed. Then one has by the above consideration for any admitted target space $A^{s_2}_{p_2q_2}(\Omega)$ that $g_k(\mathrm{id}) \ge c\, k^{-s_1/p_1}$ for some $c > 0$ and all $k \in \mathbb{N}$. In other words, in the above framework there remains an *information uncertainty* how large the target space $A^{s_2}_{p_2q_2}(\Omega)$ may be chosen. But it comes out that this a general phenomenon as long as one deals with deterministic sampling in the context of $\{A^{s_1}_{p_1q_1}(\Omega), A^{s_2}_{p_2q_2}(\Omega)\} \subset D'(\Omega)$. One may assume $A = B$.

Let Ω be an arbitrary domain (open set) in $\mathbb{R}^n$. Let $s \in \mathbb{R}$, $t \in \mathbb{R}$ and $0 < p, q, u, v \le \infty$. Then we introduce the *weak sampling numbers* g^k, $k \in \mathbb{N}$,

$$g^k\big(B^s_{pq}(\Omega), B^t_{uv}(\Omega)\big) = \inf\big[\sup\big\{\|f\,|B^t_{uv}(\Omega)\| : \|f\,|B^s_{pq}(\Omega)\| \le 1,\ f \in D(\Omega\setminus\Gamma)\big\}\big], \tag{4.37}$$

where the infimum is taken over all

$$\Gamma = \big\{x^1, \dots, x^k\big\} \subset \Omega, \tag{4.38}$$

(admitting $g^k = \infty$). If, in addition, Ω is bounded,

$$B^s_{pq}(\Omega) \hookrightarrow C(\Omega), \quad \text{and} \quad \mathrm{id}\colon B^s_{pq}(\Omega) \hookrightarrow B^t_{uv}(\Omega) \tag{4.39}$$

compact, then it follows from Definition 4.1 (i) and Proposition 4.3 that

$$0 < g^k\big(B^s_{pq}(\Omega), B^t_{uv}(\Omega)\big) \le g_k(\mathrm{id}) < \infty. \tag{4.40}$$

Definition 4.5. Let Ω be an arbitrary domain in $\mathbb{R}^n$. Let $s \in \mathbb{R}$ and $0 < p, q \le \infty$. Then the *information number* $U\big(B^s_{pq}(\Omega)\big)$ is the infimum of all numbers σ with $0 \le \sigma \le \infty$, such that for any space $B^t_{uv}(\Omega)$, where $t \in \mathbb{R}$ and $0 < u, v \le \infty$, there is a constant $c > 0$ with

$$g^k\big(B^s_{pq}(\Omega), B^t_{uv}(\Omega)\big) > c\, k^{-\sigma}, \quad k \in \mathbb{N}. \tag{4.41}$$

Remark 4.6. One can replace the B-spaces by F-spaces and A-spaces. But this does not change the situation very much and is covered by the proofs below. If $\Omega = \mathbb{R}^n$

then $g^k\big(B^s_{pq}(\mathbb{R}^n), B^t_{uv}(\mathbb{R}^n)\big) \ge c > 0$, hence $\sigma = 0$. One has no information at all, $U\big(B^s_{pq}(\mathbb{R}^n)\big) = 0$. The best outcome would be $U\big(B^s_{pq}(\Omega)\big) = \infty$, but this cannot be reached as will be seen below. This is in sharp contrast to entropy numbers and s-numbers (such as approximation numbers, Kolmogorov numbers, Gelfand numbers): In bounded Lipschitz domains Ω any decay $k^{-\sigma}$, $\sigma > 0$, can be reached for these numbers if one chooses for given $B^s_{pq}(\Omega)$ the space $B^t_{uv}(\Omega)$ in the compact embedding

$$\mathrm{id}\colon B^s_{pq}(\Omega) \hookrightarrow B^t_{uv}(\Omega) \tag{4.42}$$

appropriately.

In what follows we choose $s > n\big(\frac{1}{p} - 1\big)_+$. This is convenient, although not really necessary. It ensures $B^s_{pq}(\Omega) \hookrightarrow L^{\mathrm{loc}}_1(\Omega)$, [T01, Theorem 11.2, p. 169], and according to Theorem 1.7 one does not need moment conditions in atomic expansions.

Theorem 4.7. *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $0 < p, q \le \infty$ and $s > n\big(\frac{1}{p} - 1\big)_+$. For any $0 < u, v \le \infty$ and $t \in \mathbb{R}$ there is a constant $c > 0$ such that*

$$g^k\big(B^s_{pq}(\Omega), B^t_{uv}(\Omega)\big) \ge c\, k^{-s/n}, \quad k \in \mathbb{N}. \tag{4.43}$$

Furthermore,

$$U\big(B^s_{pq}(\Omega)\big) \le s/n. \tag{4.44}$$

Proof. First we remark that (4.44) follows from (4.43) and Definition 4.5. We prove (4.43). Let Ω_0 and Ω_1 be two concentric open cubes with $\overline{\Omega_0} \subset \Omega_1$ and $\overline{\Omega_1} \subset \Omega$. Let $\varphi \in D(\mathbb{R}^n)$ be a bump function with support near the origin, $\varphi \ge 0$, and $\varphi(0) = 1$. Let $k = 2^{Jn}$ and $\Gamma = \{z^1, \ldots, z^k\} \subset \Omega$. We may assume that there are points $x^j \in \Omega_0$ with $j = 1, \ldots, 2^{Jn}$, such that the functions $\varphi\big(2^J(x - x^j)\big)$ have pairwise disjoint supports and

$$\begin{aligned} f(x) &= 2^{-Js} \sum_{j=1}^{2^{Jn}} \varphi\big(2^J(x - x^j)\big) \\ &= 2^{-Jn/p} \sum_{j=1}^{2^{Jn}} 2^{-J(s-\frac{n}{p})}\, \varphi\big(2^J(x - x^j)\big) \in D(\Omega_0 \setminus \Gamma). \end{aligned} \tag{4.45}$$

If follows from Theorem 1.7 that (4.45) is a representation of f by normalised atoms in $B^s_{pq}(\mathbb{R}^n)$ (no moment conditions are needed). One has

$$\|f \,|B^s_{pq}(\Omega)\| \le \|f \,|B^s_{pq}(\mathbb{R}^n)\| \le c\, 2^{-Jn/p}\Big(\sum_{j=1}^{2^{Jn}} 1\Big)^{1/p} = c. \tag{4.46}$$

Let $K \in D(\mathbb{R}^n)$ be a non-negative function with support near the origin, $K(0) = 1$ (one may choose $K = \varphi$). Then

$$K(1, f)(x) = \int_{\mathbb{R}^n} K(y - x)\, f(y)\, \mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{4.47}$$

can be considered as the starting term of the characterisation of $B^t_{uv}(\mathbb{R}^n)$ in terms of local means, [T06, Section 1.4, Theorem 1.10, pp. 9/10] (no moment conditions are required). With f as in (4.45) we may assume $K(1, f) \in D(\Omega_1)$ and that for some $c > 0$,

$$K(1, f)(x) \geq c\, 2^{-Js} \quad \text{if} \quad x \in \Omega_0. \tag{4.48}$$

Then one obtains from [T06, Theorem 1.10] that

$$\|f \,|B^t_{uv}(\Omega)\| \geq c\, \|K(1, f)\,|L_u(\Omega)\| \geq c'\, 2^{-Js}. \tag{4.49}$$

Now (4.43), and hence also (4.44), follow from (4.46), (4.49) and card $\Gamma \leq 2^{Jn} = k$. □

Remark 4.8. The constant $c > 0$ in (4.43) depends on the chosen equivalent quasi-norms in $B^s_{pq}(\Omega)$ and $B^t_{uv}(\Omega)$. But one can say a little bit more. For given domain Ω and fixed quasi-norm in $B^s_{pq}(\Omega)$ we furnish the spaces $B^t_{uv}(\mathbb{R}^n)$ and, by restriction, the spaces $B^t_{uv}(\Omega)$ with quasi-norms in term of local means according to [T06, Theorem 1.10, p. 10] where (4.47) is the starting term. Then c and c' in (4.49) are independent of $B^t_{uv}(\Omega)$. Now $c = c\big(B^s_{pq}(\Omega)\big) > 0$ in (4.43) can also be chosen independently of the target spaces $B^t_{uv}(\Omega)$. In other words, the unavoidable information uncertainty cannot be circumvented by choosing target spaces $B^t_{uv}(\Omega)$ with a low smoothness, $t \to -\infty$, or a small $u > 0$ in the integrability $L_u(\Omega)$.

Remark 4.9. It follows from Theorem 4.7 that one has (4.33) also in the remaining part in Figure 4.1, p. 178 with $s_2 < 0$.

Remark 4.10. The recent work of S. Heinrich, [Hei08a], [Hei08b], [Hei09] deals with randomized linear sampling numbers for maps between (classical and fractional) Sobolev spaces and Besov spaces where, via duality, the target spaces may have negative smoothness. It comes out that randomised sampling numbers may be better than deterministic sampling numbers. In particular there is no counterpart of Theorem 4.7 for randomised sampling. As for the general background of randomised sampling one may consult [NoW08].

4.2 Sampling on intervals

4.2.1 Sampling in $A^s_{pq}(I)$

In this chapter we mainly with deal applications of Faber bases to sampling numbers. We always assume that the source spaces admit Faber expansions. This restricts the smoothness parameter s and the integrability parameter p. But otherwise one obtains (almost) final assertions for all admissible target spaces and explicit constructions (universal algorithms) for optimal sampling operators. In this Section 4.2 we specify the underlying domain by the unit interval $\Omega = I = (0, 1)$ in $\mathbb{R}$. According to

Definition 4.1 one can deal with sampling numbers $g_k(\mathrm{id})$ and $g_k^{\mathrm{lin}}(\mathrm{id})$ with id as in (4.1), (4.2) either in the framework $\mathbf{M}(I)$ of measurable functions in I, or in the framework $D'(I)$ of distributions. In this Section 4.2.1 we study sampling numbers in the context of $\mathbf{M}(I)$ and shift corresponding considerations in the context of $D'(I)$ to the next Section 4.2.2.

It seems to be reasonable to recall a few previous definitions and assertions. Let $I = (0,1)$ be the unit interval on $\mathbb{R}$ and let $\mathbf{M}(I)$ be the collection of all equivalence classes of Lebesgue almost everywhere finite complex-valued functions in I, furnished with the *convergence in measure*. This specifies $\mathbf{M}(\Omega)$ in Section 4.1.1 to $\Omega = I$. More details may be found in Section 1.1.8. On this basis we introduced in Definition 1.33 (iii) the spaces $\mathbf{A}^s_{pq}(I)$ with $\mathbf{A} \in \{\mathbf{B}, \mathbf{F}\}$. We summarise a few notation and assertions needed later on. Let again

$$\big(\Delta^1_h f\big)(x) = f(x+h) - f(x), \quad \big(\Delta^{l+1}_h f\big)(x) = \Delta^l_h \big(\Delta^1_h f\big)(x), \tag{4.50}$$

with $x \in \mathbb{R}$, $h \in \mathbb{R}$, and $l \in \mathbb{N}$ be the usual differences in $\mathbb{R}$ and

$$\big(\Delta^l_{h,I} f\big)(x) = \begin{cases} \Delta^l_h f(x) & \text{if } x + kh \in I \text{ for } k = 0, \ldots, l, \\ 0 & \text{otherwise,} \end{cases} \tag{4.51}$$

be the related restrictions to I. According to Theorem 1.35 the space $\mathbf{B}^s_{pq}(I)$ with $s > 0$, $0 < p, q \le \infty$ is the collection of all $f \in L_p(I)$ such that

$$\|f \,|\mathbf{B}^s_{pq}(I)\|_l = \|f \,|L_p(I)\| + \Big(\int_0^1 t^{-sq} \sup_{|h| \le t} \|\Delta^l_{h,I} f \,|L_p(I)\|^q \, \frac{\mathrm{d}t}{t} \Big)^{1/q} < \infty \tag{4.52}$$

where $s < l \in \mathbb{N}$ (equivalent quasi-norms). As far as corresponding spaces $\mathbf{F}^s_{pq}(I)$ are concerned we mention that

$$\mathbf{B}^s_{p,\min(p,q)}(I) \hookrightarrow \mathbf{F}^s_{pq}(I) \hookrightarrow \mathbf{B}^s_{p,\max(p,q)}(I), \tag{4.53}$$

$s > 0$, $0 < p < \infty$, $0 < q \le \infty$. This follows from (1.143) by restriction. By (1.138) one has

$$\mathbf{B}^s_{pq}(I) = B^s_{pq}(I) \quad \text{if } 0 < p, q \le \infty, \ s > \big(\tfrac{1}{p} - 1\big)_+, \tag{4.54}$$

appropriately interpreted. In particular,

$$\mathbf{B}^s_{pq}(I) = B^s_{pq}(I) \hookrightarrow C(I) \quad \text{if } 0 < p, q \le \infty, \quad s > 1/p, \tag{4.55}$$

as a special case of (4.15). Here (4.55) must be interpreted as in (4.2). Finally we recall the Faber expansion for the spaces

$$B^s_{pq}(I), \quad 0 < p, q \le \infty, \quad \tfrac{1}{p} < s < 1 + \min\big(\tfrac{1}{p}, 1\big), \tag{4.56}$$

adapted to our needs, Figure 3.1, p. 127. Let

$$\big\{ v_0, v_1, v_{jm} : j \in \mathbb{N}_0;\ m = 0, \ldots, 2^j - 1 \big\} \tag{4.57}$$

with

$$v_0(x) = 1 - x, \quad v_1(x) = x, \quad \text{where } 0 \le x \le 1, \tag{4.58}$$

and

$$v_{jm}(x) = \begin{cases} 2^{j+1}\big(x - 2^{-j}m\big) & \text{if } 2^{-j}m \le x < 2^{-j}m + 2^{-j-1}, \\ 2^{j+1}\big(2^{-j}(m+1) - x\big) & \text{if } 2^{-j}m + 2^{-j-1} \le x < 2^{-j}(m+1), \\ 0 & \text{otherwise,} \end{cases} \tag{4.59}$$

$0 \le x \le 1$, be the Faber system in I according to (3.1)–(3.3), Figure 2.1, p. 64. Let

$$\lambda_{jm}(f) = -\tfrac{1}{2}\big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m), \quad j \in \mathbb{N}_0;\ m = 0, \ldots, 2^j - 1. \tag{4.60}$$

Then $f \in B^s_{pq}(I)$ with p, q, s as in (4.56) can be uniquely represented as

$$f(x) = f(0)\, v_0(x) + f(1)\, v_1(x) + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} \lambda_{jm}(f)\, v_{jm}(x), \quad x \in I, \tag{4.61}$$

with

$$\|f \,|B^s_{pq}(I)\| \sim |f(0)| + |f(1)| + \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{1}{p})q} \Big(\sum_{m=0}^{2^j-1} |\lambda_{jm}(f)|^p\Big)^{q/p}\Big)^{1/q} \tag{4.62}$$

(with the usual modifications if $p = \infty$ and/or $q = \infty$). This is a reformulation of Theorem 3.1. The beginning

$$S^J f(x) = f(0)\, v_0(x) + f(1)\, v_1(x) + \sum_{j=0}^{J} \sum_{m=0}^{2^j-1} \lambda_{jm}(f)\, v_{jm}(x), \quad x \in I, \tag{4.63}$$

$J \in \mathbb{N}$, of (4.61) has $1 + 2^{J+1}$ terms evaluating f at the points $2^{-J-1}l$ with $l = 0, \ldots, 2^{J+1}$. Recall that $\mathbf{A}^s_{pq}(I)$ means either $\mathbf{B}^s_{pq}(I)$ or $\mathbf{F}^s_{pq}(I)$. The sampling numbers $g_k(\mathrm{id})$ and $g^{\mathrm{lin}}_k(\mathrm{id})$ have the same meaning as in Definition 4.1. As before $a_+ = \max(a, 0)$, $a \in \mathbb{R}$. Our use of $\sim$ has been explained in (4.11), (4.12).

Theorem 4.11. *Let*

$$0 < p_1, q_1 \le \infty, \quad \frac{1}{p_1} < s_1 < 1 + \min\Big(\frac{1}{p_1}, 1\Big), \tag{4.64}$$

($p_1 < \infty$ for F-spaces), and let either

$$\mathrm{id}\colon \mathbf{A}^{s_1}_{p_1q_1}(I) \hookrightarrow \mathbf{A}^{s_2}_{p_2q_2}(I), \quad 0 < p_2, q_2 \le \infty, 0 < s_2 < s_1 - \big(\tfrac{1}{p_1} - \tfrac{1}{p_2}\big)_+, \tag{4.65}$$

($p_2 < \infty$ for F-spaces) or

$$\mathrm{id}\colon \mathbf{A}^{s_1}_{p_1q_1}(I) \hookrightarrow L_{p_2}(I), \quad 0 < p_2 \le \infty,\ s_2 = 0. \tag{4.66}$$

Then

$$g_k(\mathrm{id}) \sim g_k^{\mathrm{lin}}(\mathrm{id}) \sim k^{-(s_1-s_2)+(\frac{1}{p_1}-\frac{1}{p_2})_+}, \quad k \in \mathbb{N}, \tag{4.67}$$

Figure 4.2. Further, for any admitted couple $\{\mathbf{A}^{s_1}_{p_1q_1}(I), \mathbf{A}^{s_2}_{p_2q_2}(I)\}$ *(with* $\mathbf{A}^0_{p_2q_2}(I) = L_{p_2}(I)$*) there is a constant* $c > 0$ *such that*

$$\|\mathrm{id} - S^J\| \le c\, g_{1+2^{J+1}}(\mathrm{id}), \quad J \in \mathbb{N}_0, \tag{4.68}$$

where S^J *is given by* (4.60), (4.63) (*universal order-optimal constructive algorithms*).

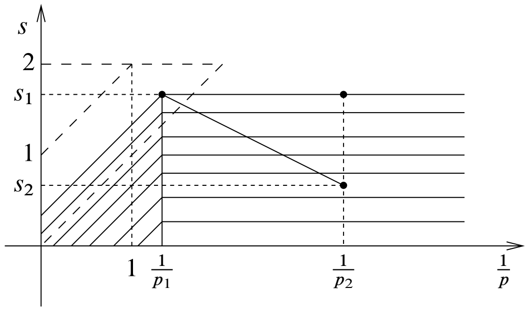

Figure 4.2. Sampling in $\mathbf{A}^s_{pq}$.

Proof. Step 1. By (4.53) it is sufficient to deal with $\mathbf{A} = \mathbf{B}$ both for the source spaces and the target spaces with $s_2 > 0$. First we prove that

$$g_k^{\mathrm{lin}}\big(\mathrm{id}\colon B^{s_1}_{p_1q_1}(I) \hookrightarrow L_{p_2}(I)\big) \le c\, k^{-s_1+(\frac{1}{p_1}-\frac{1}{p_2})_+}, \quad k \in \mathbb{N}. \tag{4.69}$$

By (4.61), (4.63) one has for $p_2 \ge 1$,

$$\begin{aligned} \|f - S^J f\,|L_{p_2}(I)\| &\le \sum_{j>J} \Big\| \sum_{m=0}^{2^j-1} \lambda_{jm}(f)\, v_{jm}\,|L_{p_2}(I)\Big\| \\ &\le c \sum_{j>J} 2^{-j/p_2} \Big(\sum_{m=0}^{2^j-1} |\lambda_{jm}(f)|^{p_2}\Big)^{1/p_2} \end{aligned} \tag{4.70}$$

and for $p_2 < 1$,

$$\|f - S^J f\,|L_{p_2}(I)\| \le c \Big(\sum_{j>J} 2^{-j} \sum_{m=0}^{2^j-1} |\lambda_{jm}(f)|^{p_2}\Big)^{1/p_2}. \tag{4.71}$$

If $p_2 \le p_1$ then one obtains from (4.62), $s_1 > 1/p_1 \ge 0$, and (4.70), (4.71) (applied to p_1) that

$$\|f - S^J f\,|L_{p_2}(I)\| \le \|f - S^J f\,|L_{p_1}(I)\| \le c\, 2^{-Js_1}\, \|f\,|B^{s_1}_{p_1q_1}(I)\|. \tag{4.72}$$

If $p_2 > p_1$ and $p_2 \geq 1$ then it follows from (4.70) that

$$\begin{aligned}\|f - S^J f \,|L_{p_2}(I)\| &\leq c \sum_{j>J} 2^{-j(s_1 - \frac{1}{p_1} + \frac{1}{p_2})}\, 2^{j(s_1 - \frac{1}{p_1})} \Big(\sum_{m=0}^{2^j - 1} |\lambda_{jm}(f)|^{p_1} \Big)^{1/p_1} \\ &\leq c'\, 2^{-J(s_1 - \frac{1}{p_1} + \frac{1}{p_2})} \, \|f \,|B^{s_1}_{p_1 q_1}(I)\|,\end{aligned} \tag{4.73}$$

where we used that $s_1 - \frac{1}{p_1} + \frac{1}{p_2} > 0$ and (4.62). If $p_1 < p_2 < 1$ then we rely on (4.71). Now one obtains (4.69) from (4.72), (4.73) and the comments about $S^J f$ after (4.63).

Step 2. If $p_1 = p_2$ and $0 < s_2 < s_1$ then it follows from Step 1 and the real interpolation formula (1.140) with $\Omega = I$ and p_1 in place of p that

$$g^{\mathrm{lin}}_k \big(\mathrm{id}\colon B^{s_1}_{p_1 q_1}(I) \hookrightarrow \mathbf{B}^{s_2}_{p_1 q_2}(I)\big) \leq c\, k^{-(s_1 - s_2)}, \quad k \in \mathbb{N}. \tag{4.74}$$

If $p_2 > p_1$ then we apply first the limiting embedding

$$\mathrm{id}\colon B^{s_1}_{p_1 q_1}(I) \hookrightarrow B^{s_3}_{p_2 q_1}(I), \quad s_3 = s_1 - \tfrac{1}{p_1} + \tfrac{1}{p_2}, \tag{4.75}$$

and afterwards (4.74) with $B^{s_3}_{p_2 q_1}(I)$ in place of $B^{s_1}_{p_1 q_1}(I)$ and $\mathbf{B}^{s_2}_{p_2 q_2}(I)$ in place of $\mathbf{B}^{s_2}_{p_1 q_2}(I)$ (similarly as indicated in Figure 4.1, p. 178). This results in

$$g^{\mathrm{lin}}_k \big(\mathrm{id}\colon B^{s_1}_{p_1 q_1}(I) \hookrightarrow \mathbf{B}^{s_2}_{p_2 q_2}(I)\big) \leq c\, k^{-s_1 + s_2 + \frac{1}{p_1} - \frac{1}{p_2}}, \quad k \in \mathbb{N}. \tag{4.76}$$

If $p_2 < p_1$ then it follows from (4.52) that

$$\mathrm{id}\colon \mathbf{B}^{s_2}_{p_1 q_2}(I) \hookrightarrow \mathbf{B}^{s_2}_{p_2 q_2}(I). \tag{4.77}$$

Then one obtains by (4.74) that

$$g^{\mathrm{lin}}_k \big(\mathrm{id}\colon B^{s_1}_{p_1 q_1}(I) \hookrightarrow \mathbf{B}^{s_2}_{p_2 q_2}(I)\big) \leq c\, k^{-(s_1 - s_2)}, \quad k \in \mathbb{N}. \tag{4.78}$$

This proves

$$g^{\mathrm{lin}}_k (\mathrm{id}) \leq c\, k^{-(s_1 - s_2) + (\frac{1}{p_1} - \frac{1}{p_2})_+}, \quad k \in \mathbb{N}, \tag{4.79}$$

in all cases of the theorem.

Step 3. If the target space is either $\mathbf{B}^0_{p_2 q_2}(I) = L_{p_2}(I)$, $0 < p_2 \leq \infty$ (according to our convention) or

$$\mathbf{B}^{s_2}_{p_2 q_2}(I) = B^{s_2}_{p_2 q_2}(I) \quad \text{with } \big(\tfrac{1}{p_2} - 1\big)_+ < s_2 < s_1 - \big(\tfrac{1}{p_1} - \tfrac{1}{p_2}\big)_+ \tag{4.80}$$

then

$$g_k \big(\mathrm{id}\colon B^{s_1}_{p_1 q_1}(I) \hookrightarrow \mathbf{B}^{s_2}_{p_2 q_2}(I)\big) \sim k^{-(s_1 - s_2) + (\frac{1}{p_1} - \frac{1}{p_2})_+} \tag{4.81}$$

follows from (4.18), (4.19) and (4.20)–(4.23). Now one obtains the theorem in these cases from (4.79), (4.81) and $g_k(\mathrm{id}) \leq g^{\mathrm{lin}}_k(\mathrm{id})$.

Step 4. It remains to prove that

$$g_k\big(\mathrm{id}\colon B^{s_1}_{p_1q_1}(I) \hookrightarrow \mathbf{B}^{s_2}_{p_2q_2}(I)\big) \ge c\, k^{-(s_1-s_2)}, \quad k \in \mathbb{N}, \tag{4.82}$$

for some $c > 0$ if $p_2 \le p_1$, $0 < s_2 \le \frac{1}{p_2} - 1$, and a counterpart for those target spaces $\mathbf{B}^{s_2}_{p_2q_2}(I)$ with $p_2 > p_1$ not covered so far. Both spaces in (4.82) can be quasi-normed by (4.52) with $0 < s_2 < s_1 < l \in \mathbb{N}$. If $0 < \theta < 1$ and

$$s = (1-\theta)s_1 + \theta s_2, \quad \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \tag{4.83}$$

then it follows from Hölder's inequality that

$$\|f\,|\mathbf{B}^s_{pq}(I)\| \le c\, \|f\,|B^{s_1}_{p_1q_1}(I)\|^{1-\theta}\, \|f\,|\mathbf{B}^{s_2}_{p_2q_2}(I)\|^{\theta}, \tag{4.84}$$

Figure 4.2, p. 186, $f \in B^{s_1}_{p_1q_1}(I)$. One obtains by Proposition 4.3 that

$$g_k\big(\mathrm{id}\colon B^{s_1}_{p_1q_1}(I) \hookrightarrow \mathbf{B}^{s}_{pq}(I)\big) \le c\, g_k^{\theta}\big(\mathrm{id}\colon B^{s_1}_{p_1q_1}(I) \hookrightarrow \mathbf{B}^{s_2}_{p_2q_2}(I)\big). \tag{4.85}$$

Now one can prove (4.82) by contradiction assuming that there is a sequence of natural numbers $k_1 < k_2 < \cdots < k_j \to \infty$ if $j \to \infty$ such that

$$g_{k_j}\big(\mathrm{id}\colon B^{s_1}_{p_1q_1}(I) \hookrightarrow \mathbf{B}^{s_2}_{p_2q_2}(I)\big) \le \varepsilon_j\, k_j^{-(s_1-s_2)}, \quad j \in \mathbb{N}, \tag{4.86}$$

with $\varepsilon_j \to 0$ if $j \to \infty$. We choose $\theta > 0$ small such that $s > 1/p$. Then it follows from (4.83), (4.85), (4.86) that

$$g_{k_j}\big(\mathrm{id}\colon B^{s_1}_{p_1q_1}(I) \hookrightarrow B^{s}_{pq}(I)\big) \le c\, \varepsilon_j^{\theta}\, k_j^{-(s_1-s)}, \quad j \in \mathbb{N}. \tag{4.87}$$

But this contradicts (4.23). One can argue similarly for the remaining cases with $p_2 > p$.

Step 5. The assertion (4.68) follows now from the above construction resulting in (4.79) and (4.67). ☐

Remark 4.12. If the parameters p_1, q_1, s_1 are restricted by (4.64) then Theorem 4.11 gives a final answer about sampling in the framework of spaces of measurable functions $\{\mathbf{A}^{s_1}_{p_1q_1}(I), \mathbf{A}^{s_2}_{p_2q_2}(I)\} \subset \mathbf{M}(I)$ on intervals. It is desirable to modify and to extend Theorem 4.11 in several directions. First one may ask for a counterpart in the framework of distributional spaces $\{A^{s_1}_{p_1q_1}(I), A^{s_2}_{p_2q_2}(I)\} \subset D'(I)$. This will be done in Section 4.2.2 below. What about an extension of Theorem 4.11 and its distributional counterpart to arbitrary admitted source spaces $A^{s_1}_{p_1q_1}(I) = \mathbf{A}^{s_1}_{p_1q_1}(I)$ with $s_1 > 1/p_1$? The above technique would require expansions of type (4.61) with smoother functions in place of v_{jm} such that the respective coefficients $\lambda_{jm}(f)$ admit a pointwise evaluation as in (4.60). We refer to Theorem 3.40 where we outlined such a possibility. But this has not yet studied in detail. It is even more interesting to ask for higher-dimensional counterparts of Theorem 4.11 especially for distributional spaces on cubes or on bounded Lipschitz domains. We return to this point in Section 4.3 in the context of spaces with dominating mixed smoothness replacing Faber bases (4.57)–(4.59) for the interval I by corresponding Faber bases (3.57)–(3.62) on $\mathbb{Q}^2$ (as a model case).

4.2.2 Sampling in $A^s_{pq}(I)$

We ask for a counterpart of Theorem 4.11 for the usual (distributional) spaces $A^s_{pq}(I)$. Again $A \in \{B, F\}$. Then $s_2 > 0$ in Theorem 4.11 should be replaced by $s_2 \in \mathbb{R}$. Recall the well-known counterpart

$$B^s_{p,\min(p,q)}(I) \hookrightarrow F^s_{pq}(I) \hookrightarrow B^s_{p,\max(p,q)}(I) \tag{4.88}$$

of (4.53) now for $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, which again gives the possibility to restrict the considerations to the B-spaces. For the source spaces we assume as before (4.55). The identification (4.54) will be of some use for us. Furthermore we rely again on the Faber expansions (4.60), (4.61) for $f \in B^s_{pq}(I)$ with (4.56). In particular, $S^J f$ has the same meaning as in (4.63). In contrast to Theorem 4.11 we have now a curious splitting between $p_1 \ge 1$ and $p_1 < 1$.

Theorem 4.13. (i) *Let*

$$1 \le p_1 \le \infty, \quad \frac{1}{p_1} < s_1 < 1 + \frac{1}{p_1}, \quad 0 < q_1 \le \infty, \tag{4.89}$$

($p_1 < \infty$ *for* F*-spaces*) *and*

$$\mathrm{id}\colon A^{s_1}_{p_1q_1}(I) \hookrightarrow A^{s_2}_{p_2q_2}(I), \quad 0 < p_2, q_2 \le \infty, \ -\infty < s_2 < s_1 - \Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+, \tag{4.90}$$

($p_2 < \infty$ *for* F*-spaces*). *Then*

$$g_k(\mathrm{id}) \sim g_k^{\mathrm{lin}}(\mathrm{id}) \sim k^{-\sigma}, \quad k \in \mathbb{N}, \tag{4.91}$$

with

$$\sigma = \begin{cases} s_1 - s_2 - \frac{1}{p_1} + \frac{1}{p_2} & \text{if } p_2 > p_1,\ s_2 > \frac{1}{p_2} - \frac{1}{p_1}, \\ s_1 - s_2 & \text{if } p_2 \le p_1,\ s_2 > 0, \\ s_1 & \text{if } s_2 < \min\big(\frac{1}{p_2} - \frac{1}{p_1}, 0\big), \end{cases} \tag{4.92}$$

Figure 4.3, p. 190.

(ii) *Let*

$$0 < p_1 < 1, \quad \frac{1}{p_1} < s_1 < 2, \quad 0 < q_1 \le \infty, \tag{4.93}$$

Figure 4.2, p. 186, and let id *be as in* (4.90). *Then* (4.91) *holds with*

$$\sigma = \begin{cases} s_1 - s_2 - \frac{1}{p_1} + \frac{1}{p_2} & \text{if } p_2 > p_1,\ s_2 > \frac{1}{p_2} - 1, \\ s_1 - s_2 & \text{if } p_2 \le p_1,\ s_2 \ge \frac{1}{p_1} - 1, \end{cases} \tag{4.94}$$

Figure 4.3, p. 190.

(iii) *In any case covered by* (i), (ii) *there is a constant* $c > 0$ *such that*

$$\|\mathrm{id} - S^J\| \le c\, g_{1+2^{J+1}}(\mathrm{id}), \quad J \in \mathbb{N}_0, \tag{4.95}$$

where S^J *is given by* (4.60), (4.63) (*universal order-optimal constructive algorithms*).

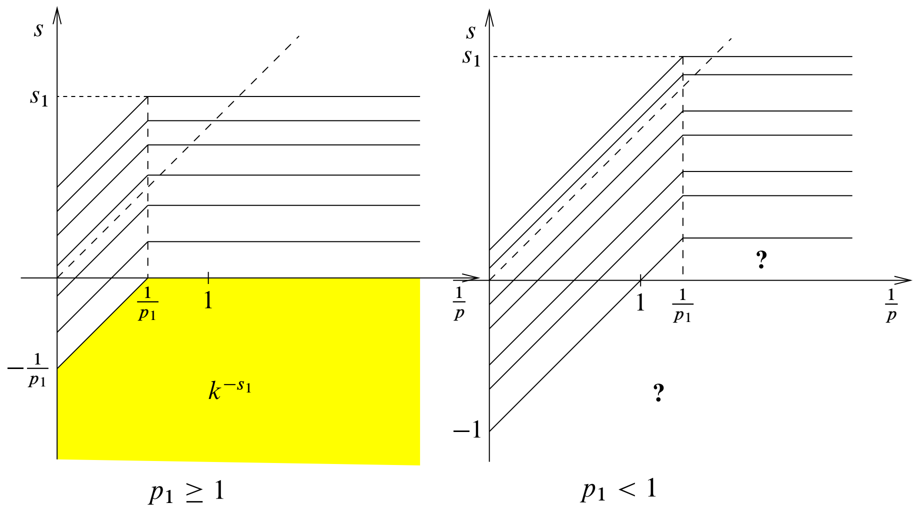

Figure 4.3. Sampling in A^s_{pq}.

Proof. *Step* 1. Let $p_1 \geq 1$. The case $p_2 \geq p_1$, $s_2 > 0$, is covered by Theorem 4.11. If $\frac{1}{p_2} - \frac{1}{p_1} < s_2 \leq 0$ then one obtains

$$g_k(\mathrm{id}) \leq g_k^{\mathrm{lin}}(\mathrm{id}) \leq c\, k^{-s_1+s_2+\frac{1}{p_1}-\frac{1}{p_2}}, \quad k \in \mathbb{N}, \tag{4.96}$$

in the same way as in (4.32). We prove the corresponding estimate from below. Let $0 < \theta < 1$,

$$p_1 < p < p_2 \leq \infty, \quad \frac{1}{p} = \frac{1-\theta}{p_2} + \frac{\theta}{p_1}, \quad s = (1-\theta)s_2 + \theta s_3, \tag{4.97}$$

and

$$s_2 - \frac{1}{p_2} = s - \frac{1}{p} = s_3 - \frac{1}{p_1}, \quad s_2 \leq 0, \quad 0 < s < s_3 < s_1. \tag{4.98}$$

This corresponds to the level line of slope 1 starting at $\big(\frac{1}{p_2}, s_2\big)$ with $\frac{1}{p_2} - \frac{1}{p_1} < s_2 \leq 0$. Recall that

$$\|f\,|B^s_{pq}(I)\| \leq c\, \|f\,|B^{s_2}_{p_2 q}(I)\|^{1-\theta}\, \|f\,|B^{s_3}_{p_1 q}(I)\|^{\theta}. \tag{4.99}$$

This follows with $\mathbb{R}$ in place of I from the original definition (1.9) and Hölder's inequality. Then one obtains (4.99) by restriction to I (having in mind that there is a universal extension from I to $\mathbb{R}$). But now one can argue in the same way as in Step 4 of the proof of Theorem 4.11 and prove the converse of (4.96) by contradiction. We assume that there is a sequence of natural numbers $k_1 < k_2 < \cdots < k_j \to \infty$ if $j \to \infty$ such that

$$g_{k_j}\big(\mathrm{id}\colon B^{s_1}_{p_1 q_1}(I) \hookrightarrow B^{s_2}_{p_2 q_2}(I)\big) \leq \varepsilon_j\, k_j^{-s_1+s_2+\frac{1}{p_1}-\frac{1}{p_2}}, \quad j \in \mathbb{N}, \tag{4.100}$$

with $\varepsilon_j \to 0$ if $j \to \infty$. We apply Proposition 4.3 and choose $2k_j$ points $\Gamma = \{x^l\}_{l=1}^{2k_j} \subset I$ such that k_j points are optimal with respect to the embedding in (4.100) and k_j points are optimal with respect to the embedding

$$\mathrm{id}\colon B^{s_1}_{p_1q_1}(I) \hookrightarrow B^{s_3}_{p_1q_2}(I). \tag{4.101}$$

Then it follows from (4.99), (4.100) and (4.96) applied to (4.101) that

$$\begin{aligned} g_{2k_j}\big(\mathrm{id}\colon B^{s_1}_{p_1q_1}(I) \hookrightarrow B^{s}_{pq_2}(I)\big) &\le c\, \varepsilon_j^{1-\theta}\, k_j^{(-s_1+s_2+\frac{1}{p_1}-\frac{1}{p_2})(1-\theta)+(-s_1+s_3)\theta} \\ &= c\, \varepsilon_j^{1-\theta}\, k_j^{-s_1+s+\frac{1}{p_1}-\frac{1}{p}}. \end{aligned} \tag{4.102}$$

But this contradicts what we already know. This proves the converse of (4.96) and hence (4.91) with the first line in (4.92). We prove (4.91) with the last line in (4.92), hence $\sigma = s_1$. Let

$$\mathrm{id}\colon B^{s_1}_{p_1q_1}(I) \hookrightarrow L_{p_1}(I). \tag{4.103}$$

Then one has by (4.19) that

$$g_k(\mathrm{id}) \sim g_k^{\mathrm{lin}}(\mathrm{id}) \sim k^{-s_1}, \quad k \in \mathbb{N}. \tag{4.104}$$

With $\big(\frac{1}{p_2}, s_2\big)$ as in the third line in (4.92) one obtains by embedding, Figure 4.3, p. 190, that

$$g_k^{\mathrm{lin}}(\mathrm{id}) \le c\, k^{-s_1}, \quad k \in \mathbb{N}. \tag{4.105}$$

The converse follows from Theorem 4.7. It remains to prove (4.91) with σ in the middle line in (4.92). Let $p_2 \le p_1$ and $s_2 > 0$. The estimate

$$g_k^{\mathrm{lin}}\big(\mathrm{id}\colon B^{s_1}_{p_1q_1}(I) \hookrightarrow B^{s_2}_{p_2q_2}(I)\big) \le c\, k^{-s_1+s_2}, \quad k \in \mathbb{N}, \tag{4.106}$$

follows from the case $p_2 = p_1$, covered by the above considerations, and the counterpart of (4.77). It remains to prove that

$$g_k\big(\mathrm{id}\colon B^{s_1}_{p_1q_1}(I) \hookrightarrow B^{s_2}_{p_2q_2}(I)\big) \ge c\, k^{-s_1+s_2}, \quad k \in \mathbb{N}, \tag{4.107}$$

for some $c > 0$ and $p_2 < p_1$, $s_2 > 0$. One can argue as in Step 4 of the proof of Theorem 4.11 where

$$\|f\,|B^{s}_{pq}(I)\| \le c\, \|f\,|B^{s_1}_{p_1q_1}(I)\|^{1-\theta}\, \|f\,|B^{s_2}_{p_2q_2}(I)\|^{\theta} \tag{4.108}$$

is the counterpart of (4.84) with (4.83). Here (4.108) can be proved similarly as (4.99). But now one is in the same situation as in Step 4 of the proof of Theorem 4.11. This proves (4.107).

Step 2. Let $p_1 < 1$. One can argue as in Step 1, but to stay within $D'(I)$ one has to replace (4.103), (4.104) by

$$\mathrm{id}\colon B^{s_1}_{p_1q_1}(I) \hookrightarrow B^{s_2}_{p_1q_2}(I), \quad \tfrac{1}{p_1} - 1 < s_2 < s_1, \tag{4.109}$$

with

$$g_k(\mathrm{id}) \sim g_k^{\mathrm{lin}}(\mathrm{id}) \sim k^{-(s_1-s_2)}, \quad k \in \mathbb{N}. \tag{4.110}$$

This follows from Theorem 4.11 and the identification (4.54).

Step 3. The assertion (4.95) follows from the above construction reducing the corresponding estimates to its counterpart (4.68). □

Remark 4.14. If $p_1 \geq 1$ then part (i) of the above theorem gives a rather final answer with exception of the borderline cases

$$s_2 = \min\Big(\frac{1}{p_2} - \frac{1}{p_1}, 0\Big). \tag{4.111}$$

Then it may happen that some powers of $\log k$ are coming in as in [Vyb07], [Vyb08a]. The outcome in case of $p_1 < 1$ is less complete. In particular there is no counterpart ensuring $g_k(\mathrm{id}) \sim k^{-s_1}$.

4.3 Sampling in spaces with dominating mixed smoothness

4.3.1 Introduction, preliminaries

We dealt in Section 4.2 with sampling numbers for spaces $\mathbf{A}^s_{pq}(I)$ and $A^s_{pq}(I)$ on intervals based on the Faber representations (4.60)–(4.62) for the source spaces. The q-index did not play any role. This is also largely the case for sampling numbers for mappings between isotropic spaces in bounded Lipschitz domains as described in Section 4.1. In this Section 4.3 we deal with sampling numbers for spaces with dominating mixed smoothness preferable of type

$$S^r_{pq}B(\mathbb{Q}^2), \quad S^r_{pq}\mathfrak{B}(\mathbb{Q}^2), \quad \text{and} \quad S^1_pW(\mathbb{Q}^2). \tag{4.112}$$

The situation is now somewhat different and the q-index plays some role.

We recall some basic definitions and assertions. Let again

$$\mathbb{Q}^2 = \big\{x = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < 1\big\} \tag{4.113}$$

be the unit square in $\mathbb{R}^2$. Let $S^r_{pq}B(\mathbb{Q}^2)$ with $r \in \mathbb{R}$ and $0 < p, q \leq \infty$ be the spaces with dominating mixed smoothness as introduced in Definition 1.56 by restriction of $S^r_{pq}B(\mathbb{R}^2)$ to $\mathbb{Q}^2$. As before the restriction of the Sobolev space $S^1_pW(\mathbb{R}^2)$, $1 < p < \infty$ according to (1.144), (1.145), to $\mathbb{Q}^2$ is denoted by $S^1_pW(\mathbb{Q}^2)$. It can be equivalently normed by (3.118), in particular,

$$\|f\,|S^1_pW(\mathbb{Q}^2)\| \sim \|f\,|L_p(\mathbb{Q}^2)\| + \sum_{l=1}^{2}\Big\|\frac{\partial f}{\partial x_l}\,|L_p(\mathbb{Q}^2)\Big\| + \Big\|\frac{\partial^2 f}{\partial x_1 \partial x_2}\,|L_p(\mathbb{Q}^2)\Big\|. \tag{4.114}$$

We adapt the previous assertions about Faber expansions for some spaces $S^r_{pq}B(\mathbb{Q}^2)$ to our later needs. Let

$$\{v_0, v_1, v_{jm} : j \in \mathbb{N}_0;\ m = 0, \ldots, 2^j - 1\} \tag{4.115}$$

be the Faber system for the unit interval $I = (0,1)$ according to (4.57)–(4.59). Let

$$\{v_{km} : k \in \mathbb{N}^2_{-1},\ m \in \mathbb{P}^F_k\} \tag{4.116}$$

be the same Faber system on $\mathbb{Q}^2$ as in (3.62) consisting of the functions

$$v_{km}(x) = \begin{cases} v_{m_1}(x_1)\, v_{m_2}(x_2) \\ \qquad \text{if } k = (-1,-1);\ m_1 \in \{0,1\},\ m_2 \in \{0,1\}, \\ v_{m_1}(x_1)\, v_{k_2 m_2}(x_2) \\ \qquad \text{if } k = (-1,k_2),\ k_2 \in \mathbb{N}_0;\ m_1 \in \{0,1\},\ m_2 = 0, \ldots, 2^{k_2} - 1, \\ v_{k_1 m_1}(x_1)\, v_{m_2}(x_2) \\ \qquad \text{if } k = (k_1,-1),\ k_1 \in \mathbb{N}_0;\ m_1 = 0, \ldots, 2^{k_1} - 1,\ m_2 \in \{0,1\}, \\ v_{k_1 m_1}(x_1)\, v_{k_2 m_2}(x_2) \\ \qquad \text{if } k \in \mathbb{N}^2_0;\ m_l = 0, \ldots, 2^{k_l} - 1;\ l = 1,2, \end{cases} \tag{4.117}$$

$x = (x_1, x_2) \in \mathbb{Q}^2$, where

$$\mathbb{P}^F_k = \{m \in \mathbb{Z}^2 \text{ with } m \text{ as in (4.117)}\}, \tag{4.118}$$

repeated here for sake of convenience. For the same reason we remind of the iterated mixed differences (3.74)–(3.77),

$$d^2_{km}(f) = f(m_1, m_2) \quad \text{if } k = (-1,-1);\ m_1 \in \{0,1\};\ m_2 \in \{0,1\}, \tag{4.119}$$

$$\begin{aligned} d^2_{km}(f) = &-\tfrac{1}{2}\Delta^2_{2^{-k_2-1},2} f\big(m_1, 2^{-k_2} m_2\big) \\ &\text{if } k = (-1,k_2),\ k_2 \in \mathbb{N}_0;\ m_1 \in \{0,1\},\ m_2 = 0, \ldots, 2^{k_2} - 1, \end{aligned} \tag{4.120}$$

$$\begin{aligned} d^2_{km}(f) = &-\tfrac{1}{2}\Delta^2_{2^{-k_1-1},1} f\big(2^{-k_1} m_1, m_2\big) \\ &\text{if } k = (k_1,-1),\ k_1 \in \mathbb{N}_0;\ m_2 \in \{0,1\},\ m_1 = 0, \ldots, 2^{k_1} - 1, \end{aligned} \tag{4.121}$$

$$\begin{aligned} d^2_{km}(f) = &\tfrac{1}{4}\Delta^{2,2}_{2^{-k_1-1},2^{-k_2-1}} f\big(2^{-k_1} m_1, 2^{-k_2} m_2\big) \\ &\text{if } k \in \mathbb{N}^2_0;\ m_l = 0, \ldots, 2^{k_l} - 1; l = 1,2, \end{aligned} \tag{4.122}$$

evaluating $f \in C(\mathbb{Q}^2)$ at the indicated lattice points in $\mathbb{Q}^2$. Let

$$\begin{cases} \text{either} & p = q = \infty,\ 0 < r < 1, \\ \text{or} & 0 < p,q < \infty,\ \frac{1}{p} < r < 1 + \min\big(\frac{1}{p}, 1\big), \end{cases} \tag{4.123}$$

Figure 3.2, p. 156. Then $f \in S^r_{pq}B(\mathbb{Q}^2)$ can be uniquely represented as

$$f = \sum_{k\in\mathbb{N}^2_{-1}} \sum_{m\in\mathbb{P}^F_k} d^2_{km}(f)\, v_{km} \tag{4.124}$$

where

$$\|f\,|S^r_{pq}B(\mathbb{Q}^2)\| \sim \Big(\sum_{k\in\mathbb{N}^2_{-1}} 2^{(k_1+k_2)(r-\frac{1}{p})q} \Big(\sum_{m\in\mathbb{P}^F_k} |d^2_{km}(f)|^p \Big)^{q/p} \Big)^{1/q} \tag{4.125}$$

with the usual modifications if $p = q = \infty$. This is a reformulation of corresponding assertions in Theorem 3.13 (i), covering $p = q = \infty$, and in Theorem 3.16. In Definition 3.24 we used expansions of type (4.124) to introduce spaces $S^r_{pq}\mathfrak{B}(\mathbb{Q}^2)$ with p, q, r as in (3.200) in the framework of measurable functions $\mathbf{M}(\mathbb{Q}^2)$. These spaces are mainly of interest as target spaces. This will be done in Section 4.3.3. At this moment we use this possibility to complement the above spaces $S^r_{pq}B(\mathbb{Q}^2)$ in their capacity as source spaces: Let

$$0 < p, q \le \infty, \qquad \frac{1}{p} < r < 1 + \min\Big(\frac{1}{p}, 1\Big). \tag{4.126}$$

Then $S^r_{pq}\mathfrak{B}(\mathbb{Q}^2)$ is the collection of all $f \in C(\mathbb{Q}^2)$ (or likewise $f \in L_1(\mathbb{Q}^2)$ or $f \in \mathbf{M}(\mathbb{Q}^2)$) which can be represented by (4.124) with

$$\|f\,|S^r_{pq}\mathfrak{B}(\mathbb{Q}^2)\| = \Big(\sum_{k\in\mathbb{N}^2_{-1}} 2^{(k_1+k_2)(r-\frac{1}{p})q} \Big(\sum_{m\in\mathbb{P}^F_k} |d^2_{km}(f)|^p \Big)^{q/p} \Big)^{1/q} < \infty \tag{4.127}$$

with the usual modifications if $p = \infty$ and/or $q = \infty$. For details we refer to Sections 3.3.1, 3.3.2 and in particular to Theorem 3.26. Of course,

$$S^r_{pq}\mathfrak{B}(\mathbb{Q}^2) = S^r_{pq}B(\mathbb{Q}^2) \quad \text{for } p, q, r \text{ as in (4.123)}, \tag{4.128}$$

Theorem 3.26 (ii). In other words, at this moment we complement the above spaces $S^r_{pq}B(\mathbb{Q}^2)$ by corresponding spaces with $p < q = \infty$ or $q < p = \infty$, where we do not know whether (4.128) remains valid (but this might be the case).

In what follows we are interested in sampling numbers $g_k(\mathrm{id})$ and $g^{\mathrm{lin}}_k(\mathrm{id})$ according to Definition 4.1 for compact embeddings,

$$\mathrm{id}\colon S^r_{pq}\mathfrak{B}(\mathbb{Q}^2) \hookrightarrow L_u(\mathbb{Q}^2), \tag{4.129}$$

p, q, r as above and $0 < u \le \infty$ in the framework of $\mathbf{M}(\mathbb{Q}^2)$. As for $\mathbf{M}(\mathbb{Q}^2)$ one may consult Section 3.3.1 with a reference to Section 1.1.8.

Sampling numbers for periodic spaces of type $S^r_{pq}B(\mathbb{T}^n)$, $S^r_pW(\mathbb{T}^n)$ and also $S^r_{pq}F(\mathbb{T}^n)$ on the n-torus $\mathbb{T}^n$ as source spaces and preferably $L_p(\mathbb{T}^n)$ as target spaces attracted some attention. This goes back to V. N. Temlyakov, [Tem85], his book

[Tem93] and the literature mentioned there. More recent results may be found in [Sic06], [SiU07], [Ull08]. By rule of thumb assertions for periodic spaces have non-periodic counterparts. The justification given in [SlJ94, Section 2.12] eliminates the boundary values. But this does not fit in our scheme. Boundary data will play a growing role in what follows. In the context of non-periodic spaces we refer also to [Tem93, IV, §4] dealing with cubature formulas. In this sense we compare later on a few results in the above-mentioned papers with what comes out by our approach. Recent results about approximation numbers and sampling numbers in the non-periodic case may be found in [HaS09], [SiU09].

4.3.2 Main assertions

We described in Section 4.1.2 our approach to the theory of sampling numbers for isotropic spaces in bounded Lipschitz domains Ω in $\mathbb{R}^n$ with (4.18), (4.19) as first typical results in the framework of $\mathbf{M}(\Omega)$. What follows might be considered as the counterpart of this approach in the context of some spaces with dominating mixed smoothness in $\mathbb{Q}^2$. In Section 4.3.1 we collected some background information and repeated a few relevant assertions. Recall that $a_+ = \max(a, 0)$ for $a \in \mathbb{R}$. Let again $\mathbb{Q}^2$ be the unit square (4.113) in $\mathbb{R}^2$.

Theorem 4.15. *Let*

$$0 < p, q \le \infty, \quad \frac{1}{p} < r < 1 + \min\Big(\frac{1}{p}, 1\Big), \quad 0 < u \le \infty. \tag{4.130}$$

Then the embedding

$$\mathrm{id}\colon S^r_{pq}\mathfrak{B}(\mathbb{Q}^2) \hookrightarrow L_u(\mathbb{Q}^2), \tag{4.131}$$

considered in the framework of $\mathbf{M}(\mathbb{Q}^2)$*, is compact. Furthermore*

$$\begin{aligned} c_1\, l^{-r+\frac{1}{p}-\frac{1}{u}} (\log l)^{(\frac{1}{u}-\frac{1}{q})_+} &\le g_l(\mathrm{id}) \le g_l^{\mathrm{lin}}(\mathrm{id}) \\ &\le c_2 \Big(\frac{l}{\log l}\Big)^{-r+\frac{1}{p}-\frac{1}{u}} \begin{cases} (\log l)^{(1-\frac{1}{q})_+} & \text{if } p \le u,\ u \ge 1, \\ (\log l)^{(\frac{1}{u}-\frac{1}{q})_+} & \text{if } p \le u < 1, \end{cases} \end{aligned} \tag{4.132}$$

and

$$\begin{aligned} c_1\, l^{-r} (\log l)^{(1-\frac{1}{q})_+} &\le g_l(\mathrm{id}) \le g_l^{\mathrm{lin}}(\mathrm{id}) \\ &\le c_2 \Big(\frac{l}{\log l}\Big)^{-r} \begin{cases} (\log l)^{(1-\frac{1}{q})_+} & \text{if } u \le p,\ p \ge 1, \\ (\log l)^{(\frac{1}{p}-\frac{1}{q})_+} & \text{if } u \le p < 1, \end{cases} \end{aligned} \tag{4.133}$$

for some $c_1 > 0$, $c_2 > 0$ *and all* $l \in \mathbb{N}$, $l \ge 2$.

Proof. Step 1. We simplify temporarily $Q = \mathbb{Q}^2$. We split the representation (4.124) of $f \in S^r_{pq}\mathfrak{B}(Q)$ into a main part $M^K f$,

$$M^K f = \sum_{k_1+k_2 \le K} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(f)\, v_{km}, \quad K \in \mathbb{N}, \tag{4.134}$$

and a remainder part $R^K f$,

$$R^K f = \sum_{k_1+k_2 > K} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(f)\, v_{km}, \quad K \in \mathbb{N}. \tag{4.135}$$

Let first $0 < q \le p \le 1$. Then

$$\begin{aligned} &\|R^K f\,|L_p(Q)\|^p \\ &\le c \sum_{k_1+k_2>K} \sum_{m \in \mathbb{P}^F_k} 2^{-(k_1+k_2)} |d^2_{km}(f)|^p \\ &= c\, 2^{-Krp} \sum_{k_1+k_2>K} 2^{-(k_1+k_2-K)rp}\, 2^{(k_1+k_2)(r-\frac{1}{p})p} \sum_{m \in \mathbb{P}^F_k} |d^2_{km}(f)|^p. \end{aligned} \tag{4.136}$$

Then it follows from (4.127) that

$$\|R^K f\,|L_p(Q)\| \le c\, 2^{-Kr}\, \|f\,|S^r_{pq}\mathfrak{B}(Q)\|, \quad 0 < q \le p \le 1. \tag{4.137}$$

If $p \le 1$ and $p < q < \infty$ then one has by (4.136)

$$\begin{aligned} \|R^K f\,|L_p(Q)\| &\le c\, 2^{-Kr} \Big(\sum_{L>K} \sum_{k_1+k_2=L} 2^{-(k_1+k_2-K)r\frac{pq}{q-p}} \Big)^{\frac{q-p}{pq}} \|f\,|S^r_{pq}\mathfrak{B}(Q)\| \\ &\le c'\, 2^{-Kr} \Big(\sum_{L>K} 2^{-(L-K)r\frac{pq}{q-p}} (L-K+K) \Big)^{\frac{q-p}{pq}} \|f\,|S^r_{pq}\mathfrak{B}(Q)\| \\ &\le c''\, 2^{-Kr}\, K^{\frac{1}{p}-\frac{1}{q}} \|f\,|S^r_{pq}\mathfrak{B}(Q)\|. \end{aligned} \tag{4.138}$$

Similarly if $p \le 1$ and $q = \infty$. Hence one has by (4.137), (4.138) and Hölder's inequality that

$$\|R^K f\,|L_u(Q)\| \le c\, 2^{-Kr}\, K^{(\frac{1}{p}-\frac{1}{q})_+} \|f\,|S^r_{pq}\mathfrak{B}(Q)\|, \quad 0 < u \le p \le 1, \tag{4.139}$$

$0 < q \le \infty$, $K \in \mathbb{N}$. Let $p > 1$. Then

$$\begin{aligned} &\|R^K f\,|L_p(Q)\| \\ &\le c \sum_{k_1+k_2>K} 2^{-(k_1+k_2)/p} \Big(\sum_{m \in \mathbb{P}^F_k} |d^2_{km}(f)|^p \Big)^{1/p} \\ &= c\, 2^{-Kr} \sum_{k_1+k_2>K} 2^{-(k_1+k_2-K)r}\, 2^{(k_1+k_2)(r-\frac{1}{p})} \Big(\sum_{m \in \mathbb{P}^F_k} |d^2_{km}(f)|^p \Big)^{1/p} \end{aligned} \tag{4.140}$$

is the counterpart of (4.136). If $q \le 1$ then one can argue as in (4.137). If $q > 1$ then one modifies (4.138) by

$$\begin{aligned} &\|R^K f \,|L_p(Q)\| \\ &\le c\, 2^{-Kr} \Big(\sum_{L>K} \sum_{k_1+k_2=L} 2^{-(k_1+k_2-K)r\frac{q}{q-1}} \Big)^{1-\frac{1}{q}} \|f\,|S^r_{pq}\mathfrak{B}(Q)\| \\ &\le c'\, 2^{-Kr} K^{1-\frac{1}{q}} \|f\,|S^r_{pq}\mathfrak{B}(Q)\|. \end{aligned} \tag{4.141}$$

Together with Hölder's inequality one obtains that

$$\|R^K f\,|L_u(Q)\| \le c\, 2^{-Kr}\, K^{(1-\frac{1}{q})_+} \|f\,|S^r_{pq}\mathfrak{B}(Q)\|, \quad 0 < u \le p, \ p \ge 1, \tag{4.142}$$

$0 < q \le \infty$. Let $p < u \le \infty$. Then it follows from (4.127) that

$$S^r_{pq}\mathfrak{B}(Q) \hookrightarrow S^{r-\frac{1}{p}+\frac{1}{u}}_{uq}\mathfrak{B}(Q) \tag{4.143}$$

as indicated in Figure 4.4, p. 198. Now one obtains from (4.139), (4.142) that

$$\|R^K f\,|L_u(Q)\| \le c\, 2^{-K(r-\frac{1}{p}+\frac{1}{u})} \|f\,|S^r_{pq}\mathfrak{B}(Q)\| \begin{cases} K^{(\frac{1}{u}-\frac{1}{q})_+}, & p < u \le 1, \\ K^{(1-\frac{1}{q})_+}, & u > p,\ u > 1, \end{cases} \tag{4.144}$$

$0 < q \le \infty$, $K \in \mathbb{N}$.

Step 2. We deal with the main part $M^K f$ according to (4.134). If $k \in \mathbb{N}_0^2$ and $m \in \mathbb{P}^F_k$ then the differences $d^2_{km}(f)$ in (4.122) are linear combinations of values of f at nine points in Q. Similarly for $d^2_{km}(f)$ in (4.119)–(4.121). Hence one needs for $M^K f$ in (4.134) the knowledge of f in $\sim K2^K$ points in $\bar{Q}$. Let $l \in \mathbb{N}$, $l \ge 2$. If $K \in \mathbb{N}$ and $0 \le \varepsilon < 1$ such that

$$K = \log l - \log\log l + \varepsilon, \quad \text{then} \quad 2^K \sim \frac{l}{\log l}, \quad K2^K \sim l. \tag{4.145}$$

Then one obtains for id in (4.131) that

$$g^{\text{lin}}_l(\text{id}) \le \sup\big\{\|R^K f\,|L_u(Q)\| : \|f\,|S^r_{pq}\mathfrak{B}(Q)\| \le 1\big\}. \tag{4.146}$$

The right-hand sides of (4.133), (4.132) follow from (4.139), (4.142) and (4.144). In particular, id in (4.131) is compact.

Step 3. We prove the left-hand sides of (4.132), (4.133). We rely on Proposition 4.3, hence

$$g_l(\text{id}) \sim \inf\big\{\|\text{id}^\Gamma\| : \text{card}\,\Gamma \le l\big\}, \quad l \in \mathbb{N}, \tag{4.147}$$

where $\Gamma = \{x^j\}^l_{j=1} \subset Q$ and

$$\text{id}^\Gamma : \big\{f \in S^r_{pq}\mathfrak{B}(Q) : \textstyle\sum^l_{j=1} |f(x^j)| = 0\big\} \hookrightarrow L_u(Q). \tag{4.148}$$

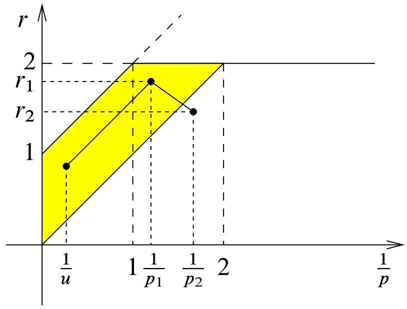

Figure 4.4. Dominating mixed smoothness.

Let l, k_1, k_2 be natural numbers with $k_1 + k_2 = l + 1$. We divide Q naturally into 2^{l+1} open rectangles Q_{km} of side-lengths 2^{-k_1} and 2^{-k_2}. Recall that supp $v_{km} = \overline{Q_{km}}$, where $k = (k_1, k_2)$. Let Γ be a set of at most 2^l points in Q and let $\mathbb{P}_k^\Gamma$ be 2^l points $m \in \mathbb{Z}^2$ such that $Q_{km} \cap \Gamma = \emptyset$. Let

$$f(x) = \sum_{m \in \mathbb{P}_k^\Gamma} v_{km}. \tag{4.149}$$

Then it follows from (4.124), (4.127) that

$$\|f \,|S^r_{pq}\mathfrak{B}(Q)\| \sim 2^{l(r-\frac{1}{p})} \Big(\sum_{m \in \mathbb{P}_k^\Gamma} 1 \Big)^{1/p} = 2^{lr}, \quad l \in \mathbb{N}, \tag{4.150}$$

and

$$\|f \,|L_u(Q)\| \sim 1. \tag{4.151}$$

This proves the left-hand side of (4.133) (with 2^l in place of l) for $q \le 1$. If one chooses only one term in (4.149), hence $f = v_{km}$, then

$$\|f \,|S^r_{pq}\mathfrak{B}(Q)\| \sim 2^{l(r-\frac{1}{p})} \quad \text{and} \quad \|f \,|L_u(Q)\| \sim 2^{-l/u}. \tag{4.152}$$

This proves the left-hand side of (4.132) for $q \le u$.

Step 4. We deal with the remaining cases for q. We replace f in (4.149) by

$$f(x) = \sum_{k_1+k_2=l+1} \sum_{m \in \mathbb{P}_k^\Gamma} v_{km}. \tag{4.153}$$

We claim that

$$\|f \,|S^r_{pq}\mathfrak{B}(Q)\| \sim l^{1/q}\, 2^{lr}, \quad \|f \,|L_u(Q)\| \sim l, \tag{4.154}$$

is the substitute of (4.150), (4.151). The first equivalence follows from (4.127). The second equivalence is obvious for $u = 1$. If $u > 1$ then it is a consequence of

$$l \sim \int_Q f(x)\,\mathrm{d}x \le \Big(\int_Q f^u(x)\,\mathrm{d}x \Big)^{1/u} \le l. \tag{4.155}$$

If $u < 1$ then it follows from

$$1 \sim \int_Q \sum_{k,m} \frac{v_{km}}{l}\,\mathrm{d}x \le \Big(\int_Q \Big(\sum_{k,m} \frac{v_{km}}{l} \Big)^u \mathrm{d}x \Big)^{1/u} = l^{-1} \Big(\int_Q f^u(x)\,\mathrm{d}x \Big)^{1/u} \le 1. \tag{4.156}$$

The left-hand side of (4.133) with $q \ge 1$ is now a consequence of (4.154). If one takes in (4.153) for any $k = (k_1, k_2)$ with $k_1 + k_2 = l + L$ for some fixed $L \in \mathbb{N}$ in place of $k_1 + k_2 = l + 1$ only one term v_{km} then one obtains that

$$\| f \,|S^r_{pq}\mathfrak{B}(Q)\| \sim l^{1/q}\, 2^{l(r-\frac{1}{p})}, \qquad \| f \,|L_u(Q)\| \ge c\, l^{1/u}\, 2^{-l/u}, \tag{4.157}$$

instead of (4.152). The first equivalence follows from (4.127). The second estimate is obvious if $u = 1$ (equivalence). If the supports $\operatorname{supp} v_{km} = \overline{Q_{km}}$ have empty intersections (or intersects only in faces) then $\| f \,|L_u(Q)\| \sim l^{1/u} 2^{-l/u}$. But one may assume that, say, $1/2$ of each Q_{km} does not intersect with any other involved $Q_{k'm'}$: Choosing $L \in \mathbb{N}$ suitably one can distribute rectangles with consecutive $k_1, k_1 - 1, \ldots, k_1 - L$ in different rectangular parts of Q. Possible intersections of rectangles Q_{km}, $k = (k_1, k_2)$, with $Q_{k'm'}$, $k' = (k_1 - L - 1, k_2)$, can cover only a small portion of Q_{km}. Then one can estimate $\| f \,|L_u(Q)\|$ as indicated in (4.157). This proves the left-hand side of (4.132) also in case of $q \ge u$. □

As before we concentrate our considerations on the spaces $S^r_{pq}B(\mathbb{Q}^2)$ complemented by a few limiting spaces, (4.128), and on the distinguished Sobolev spaces $S^1_p W(\mathbb{Q}^2)$, $1 < p < \infty$, having dominating mixed smoothness of order 1. One has by (4.114),

$$S^1_{p_1} W(\mathbb{Q}^2) \hookrightarrow S^1_2 W(\mathbb{Q}^2) = S^1_{2,2} B(\mathbb{Q}^2) \hookrightarrow S^1_{p_2} W(\mathbb{Q}^2) \quad \text{if } 1 < p_2 \le 2 \le p_1 < \infty \tag{4.158}$$

and by (3.93),

$$S^1_{p,\min(p,2)} B(\mathbb{Q}^2) \hookrightarrow S^1_p W(\mathbb{Q}^2) \hookrightarrow S^1_{p,\max(p,2)} B(\mathbb{Q}^2) \hookrightarrow C(\mathbb{Q}^2), \tag{4.159}$$

where $1 < p < \infty$. Both embeddings give the possibility to complement Theorem 4.15 as follows.

Corollary 4.16. *Let $1 < p < \infty$ and $0 < u \le \infty$. Then the embedding*

$$\mathrm{id}\colon\ S^1_p W(\mathbb{Q}^2) \hookrightarrow L_u(\mathbb{Q}^2), \tag{4.160}$$

considered in the framework of $\mathbf{M}(\mathbb{Q}^2)$, is compact.

(i) *Let, in addition, $u > p$. Then*

$$c_1\, l^{-1+\frac{1}{p}-\frac{1}{u}} \le g_l(\mathrm{id}) \le g_l^{\mathrm{lin}}(\mathrm{id}) \le c_2 \Big(\frac{l}{\log l}\Big)^{-1+\frac{1}{p}-\frac{1}{u}} (\log l)^{1-\frac{1}{p}} \tag{4.161}$$

for some $c_1 > 0$, $c_2 > 0$, and all $l \in \mathbb{N}$, $l \ge 2$.

(ii) *Let, in addition, $2 \le u \le p$. Then*

$$c_1\, l^{-1} (\log l)^{1/2} \le g_l(\mathrm{id}) \le g_l^{\mathrm{lin}}(\mathrm{id}) \le c_2\, l^{-1} (\log l)^{2-\frac{1}{u}} \tag{4.162}$$

for some $c_1 > 0$, $c_2 > 0$, and all $l \in \mathbb{N}$, $l \ge 2$.

(iii) *Let, in addition, $u \le 2$, $u \le p$. Then*

$$c_1\, l^{-1} (\log l)^{1/2} \le g_l(\mathrm{id}) \le g_l^{\mathrm{lin}}(\mathrm{id}) \le c_2\, l^{-1} (\log l)^{3/2} \tag{4.163}$$

for some $c_1 > 0$, $c_2 > 0$, and all $l \in \mathbb{N}$, $l \ge 2$.

Proof. Step 1. We prove the left-hand sides of (4.161)–(4.163). By (4.128), (4.123) one has $B = \mathfrak{B}$ in all cases covered by (4.159). The left-hand side of (4.161) follows from (4.132) with $r = 1$ and (4.159) with $u > q = \min(p, 2)$. Similarly one obtains the left-hand side of (4.162) from the left-hand side of (4.133) with $r = 1, q = \min(p, 2) = 2$. This applies also to the left-hand side of (4.163) if $p \ge 2$. If $u \le p < 2$ then we use (4.158) with $p_2 = p$ and what we already know.

Step 2. The right-hand side of (4.161) follows from (4.132) and the limiting embedding

$$S^1_p W(\mathbb{Q}^2) \hookrightarrow S^{1-\frac{1}{p}+\frac{1}{u}}_{up} B(\mathbb{Q}^2). \tag{4.164}$$

This has been proved recently in [HaV09] with $\mathbb{R}^2$ in place of $\mathbb{Q}^2$. Restriction to $\mathbb{Q}^2$ gives (4.164).

Step 3. For $2 \le u \le p$ we have by (4.159) that

$$S^1_p W(\mathbb{Q}^2) \hookrightarrow S^1_u W(\mathbb{Q}^2) \hookrightarrow S^1_{uu} B(\mathbb{Q}^2). \tag{4.165}$$

Then the right-hand side of (4.162) follows from (4.133) with $r = 1$ and $q = u$.

Step 4. Let $u \le p \le 2$. Then one obtains the right-hand side of (4.163) from (4.133) and

$$S^1_p W(\mathbb{Q}^2) \hookrightarrow S^1_{p,2} B(\mathbb{Q}^2) \tag{4.166}$$

according to (4.159). If $u \le 2$ and $p > 2$ then one uses (4.158) with $p_1 = p$ and what one already knows. □

Remark 4.17. We return to some of the references given at the end of Section 4.3.1 comparing them now with the above assertions. As mentioned there approximations of periodic functions belonging to spaces with dominating mixed smoothness in the n-torus $\mathbb{T}^n$ have been studied in the 1980s and 1990s by V. N. Temlyakov. We refer again to his book [Tem93] and the literature mentioned there. More recent results for linear sampling numbers $g_l^{\mathrm{lin}}(\mathrm{id})$ with

$$\mathrm{id}\colon S^r_{pq}A(\mathbb{T}^n) \hookrightarrow L_p(\mathbb{T}^n), \quad 1 \le p, q \le \infty,\ r > 1/p, \tag{4.167}$$

$A \in \{B, F, W\}$, may be found in [Sic06], [SiU07], [Ull08]. For $1 \le p \le \infty$ and $r > 1/p$ one has by [Tem93, IV, §5, Theorem 5.1] that

$$g_l^{\mathrm{lin}}\big(\mathrm{id}\colon S^r_{p\infty}B(\mathbb{T}^2) \hookrightarrow L_p(\mathbb{T}^2)\big) \le c\, l^{-r}(\log l)^{r+1}, \tag{4.168}$$

$l \in \mathbb{N}$, $l \ge 2$. This is the same estimate as in (4.132) with $\mathbb{Q}^2$ in place of $\mathbb{T}^2$ where $1 \le p = u \le \infty$, $q = \infty$ and $\frac{1}{p} < r < 1 + \frac{1}{p}$. According to [SiU07, Corollary 4, p. 403] one can generalise (4.168) for $1 < p < \infty$, $1 \le q \le \infty$, $r > 1/p$ to

$$g_l^{\mathrm{lin}}\big(\mathrm{id}\colon S^r_{pq}B(\mathbb{T}^2) \hookrightarrow L_p(\mathbb{T}^2)\big) \le c\, l^{-r}(\log l)^{r+1-\frac{1}{q}}, \tag{4.169}$$

$l \in \mathbb{N}$, $l \ge 2$ (and an n-dimensional counterpart). This is again in good agreement with (4.132) where $1 < p = u < \infty$, $1 \le q \le \infty$, $\frac{1}{p} < r < 1 + \frac{1}{p}$. Furthermore one has by [SiU07, Corollary 5, p. 404] and [Sic06, Theorem 2, p. 282] (as a special case) that

$$g_l^{\mathrm{lin}}\big(\mathrm{id}\colon S^1_pW(\mathbb{T}^2) \hookrightarrow L_p(\mathbb{T}^2)\big) \le c\, l^{-1}(\log l)^{3/2}, \tag{4.170}$$

$l \in \mathbb{N}$, $l \ge 2$, where $1 < p < \infty$. This is the same order of approximation as in (4.163) if $1 < p \le 2$ and a better estimate if $2 < p < \infty$. The last assertion is also a special case of [Ull08, Corollaries 4,5, p. 13]. The above-mentioned literature relies on the so-called *Smolyak algorithm* and on approximations in terms of the *hyperbolic cross*. We refer in addition to [ScS04] and the recent paper [SiU08]. We did not employ these techniques but the outcome is rather similar (as it should be). The above estimates from below, for example in (4.132), (4.133), apply to non-linear sampling numbers $g_l(\mathrm{id})$. In [Sic06], [SiU07], [Ull08] one finds corresponding estimates for $g_l^{\mathrm{lin}}(\mathrm{id})$ from below for periodic spaces of type $l^{-r}(\log l)^{\delta}$ with $0 < \delta < \sigma$ (where σ refers to related estimates from above) using partly [Gal01]. Both in Theorem 4.15, Corollary 4.16 and in the assertions of the just-mentioned papers there are gaps between the estimates from above and from below. In the isotropic case one has rather final assertions. One may consult Section 4.1.2, the literature mentioned there, and also the Theorems 4.11, 4.13. But it seems to be a rather tricky task to determine the exact behaviour of $g_l(\mathrm{id})$ and g_l^{lin} in case of (periodic or non-periodic) spaces with dominating mixed smoothness. The only case known to us where one has not only estimates but an equivalence is Temlyakov's assertion

$$g_l^{\mathrm{lin}}\big(\mathrm{id}\colon S^r_{2,2}B(\mathbb{T}^n) \hookrightarrow C(\mathbb{T}^n)\big) \sim l^{-r+\frac{1}{2}}(\log l)^{r(n-1)}, \tag{4.171}$$

$l \in \mathbb{N}$, $l \ge 2$ with $n \in \mathbb{N}$ and $r > 1/2$. We refer to [Tem93a, Theorem 1.1, p. 46]. This coincides with the right-hand side of (4.132) where $n = p = q = 2$, $u = \infty$.

Remark 4.18. The estimates from above for the linear sampling numbers $g_l^{\text{lin}}(\text{id})$ in [Tem93], [Sic06], [SiU07], [Ull08] for periodic spaces with dominating mixed smoothness coincide largely with corresponding assertions in the above non-periodic case (as far as they are comparable), Remark 4.17. The situation for estimates from below is somewhat different. We restrict ourselves to an example. Let

$$\text{id}^{\text{per}}\colon S^r_{pp}B(\mathbb{T}^2) \hookrightarrow L_p(\mathbb{T}^2), \quad 1 < p < \infty,\ r > 1/p. \tag{4.172}$$

Then it follows from [SiU07, Corollary 4, p. 403] that

$$c_1\left(\frac{l}{\log l}\right)^{-r}(\log l)^{(\frac{1}{2}-\frac{1}{p})_+} \le g_l^{\text{lin}}(\text{id}^{\text{per}}) \le c_2\left(\frac{l}{\log l}\right)^{-r}(\log l)^{1-\frac{1}{p}} \tag{4.173}$$

for some $c_1 > 0$, $c_2 > 0$ and all $l \in \mathbb{N}$, $l \ge 2$. Let

$$\text{id}\colon S^r_{pp}B(\mathbb{Q}^2) \hookrightarrow L_p(\mathbb{Q}^2), \quad 1 \le p \le \infty,\ 1/p < r < 1 + 1/p. \tag{4.174}$$

Then it follows from Theorem 4.15 that

$$c_1\left(\frac{l}{\log l}\right)^{-r}(\log l)^{-r+1-\frac{1}{p}} \le g_l^{\text{lin}}(\text{id}) \le c_2\left(\frac{l}{\log l}\right)^{-r}(\log l)^{1-\frac{1}{p}} \tag{4.175}$$

for some $c_1 > 0$, $c_2 > 0$ and all $l \in \mathbb{N}$, $l \ge 2$. Whereas one has the same right-hand sides in (4.173), (4.175) the corresponding left-hand sides are different. If $p > 2$ and $\frac{1}{p} < r < \frac{1}{2}$ then $-r+1-\frac{1}{p} > \frac{1}{2}-\frac{1}{p}$. In other words, the estimate from below in (4.175) is better than the corresponding estimate in (4.173). The corresponding assertions in [SiU07] are based on related estimates for approximation numbers (which are smaller than $g_l^{\text{lin}}(\text{id}^{\text{per}})$) with a reference to [Gal01]. These assertions have been improved very recently in [SiU09] by other methods dealing with the n-dimensional version of (4.174) where $1 \le p \le \infty$, $1/p < r < 2$. This is based on corresponding estimates for approximation numbers where $1 \le p \le \infty$, $r > 0$, with a reference to [HaS09]. The arguments rely on (non-periodic) splines, in particular Faber splines. It comes out that our restriction $\frac{1}{p} < r < 1 + \frac{1}{p}$ in case of $1 \le p \le \infty$ can be improved by $1/p < r < 2$ (using again second differences). Otherwise the outcome is similar as in (4.173), n-dimensional case with $\mathbb{Q}^n$ in place of $\mathbb{T}^n$.

There is a further decisive difference between the above-mentioned literature and the assertions in Theorem 4.15 and Corollary 4.16. For the target spaces $L_u(\mathbb{Q}^2)$ in (4.131), (4.160) any u with $0 < u \le \infty$ is admitted whereas in the above-mentioned literature u is usually specified by $u = p$. In connection with numerical integration the case $u = 1$ will be of great service for us later on. In Theorems 4.11, 4.13 we obtained in (4.67), (4.91) not only rather final assertions, but in (4.68), (4.95) explicit universal order-optimal algorithms. Nothing like this can be expected in Theorem 4.15 and Corollary 4.16. On the other hand we have the explicit approximating operators $M^K f$ in (4.134) resulting in the right-hand sides in Theorem 4.15 and Corollary 4.16. The function f is evaluated in lattice points. This selections of points in $\mathbb{Q}^2$ is presumably too regular to obtain optimal algorithms for $g_l^{\mathrm{lin}}(\mathrm{id})$ or even $g_l(\mathrm{id})$ as in (4.68), (4.95). However from the point of view of simplicity and the restriction to lattices, or regular grids, the operator $M^K f$ seems to be rather effective. This may justify to formulate the outcome. Let again $\{v_{km}\}$ be the Faber system (4.116)–(4.118) in $\mathbb{Q}^2$ and let $d^2_{km}(f)$ be the mixed differences according to (4.119)–(4.122). Let $l \in \mathbb{N}$, $l \geq 2$ and $K_l \in \mathbb{N}$ with

$$K_l = \log l - \log\log l + \varepsilon, \quad \text{where } 0 \leq \varepsilon < 1. \tag{4.176}$$

Let $f \in C(\mathbb{Q}^2)$. Then

$$M^{K_l} f = \sum_{k_1+k_2 \leq K_l} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(f)\, v_{km}, \quad l \in \mathbb{N},\ l \geq 2, \tag{4.177}$$

is the same operator as in (4.134) with (4.145). In particular, $M^{K_l} f$ evaluates $f \in C(\mathbb{Q}^2)$ in $\sim l$ points in $\mathbb{Q}^2$. We interpret M^{K_l} as an operator of rank $M^{K_l} \sim l$.

Corollary 4.19. (i) *Let* id *be as in* (4.130), (4.131) *and let* $g_l^{\mathrm{lin}+}(\mathrm{id})$ *be the right-hand sides of* (4.132), (4.133). *Then*

$$c_1\, g_l^{\mathrm{lin}}(\mathrm{id}) \leq \left\| \mathrm{id} - M^{K_l} : S^r_{pq}\mathfrak{B}(\mathbb{Q}^2) \hookrightarrow L_u(\mathbb{Q}^2) \right\| \leq c_2\, g_l^{\mathrm{lin}+}(\mathrm{id}) \tag{4.178}$$

for some $c_1 > 0$, $c_2 > 0$ *and all* $l \in \mathbb{N}$, $l \geq 2$.

(ii) *Let* id *be as in* (4.160) *and let* $g_l^{\mathrm{lin}+}(\mathrm{id})$ *be the right-hand sides of* (4.161)–(4.163). *Then*

$$c_1\, g_l^{\mathrm{lin}}(\mathrm{id}) \leq \left\| \mathrm{id} - M^{K_l} : S^1_p W(\mathbb{Q}^2) \hookrightarrow L_u(\mathbb{Q}^2) \right\| \leq c_2\, g_l^{\mathrm{lin}+}(\mathrm{id}) \tag{4.179}$$

for some $c_1 > 0$, $c_2 > 0$ *and all* $l \in \mathbb{N}$, $l \geq 2$.

Proof. This follows from the proofs of Theorem 4.15 and Corollary 4.16. □

4.3.3 Complements

In contrast to Theorems 4.11, 4.13 and also to corresponding assertions for sampling numbers in isotropic spaces according to Section 4.1.2 we restricted so far the target spaces in Theorem 4.15 and Corollary 4.16 to $L_u(\mathbb{Q}^2)$, $0 < u \leq \infty$. Now we replace $L_u(\mathbb{Q}^2)$ by suitable spaces with dominating mixed smoothness as introduced in Definition 3.24 and deal with compact embeddings

$$\mathrm{id}\colon S^{r_1}_{p_1 q_1}\mathfrak{B}(\mathbb{Q}^2) \hookrightarrow S^{r_2}_{p_2 q_2}\mathfrak{B}(\mathbb{Q}^2), \tag{4.180}$$

where $p_1, p_2, q_1, q_2 \in (0, \infty]$ and

$$\frac{1}{p_1} < r_1 < 1 + \min\Big(\frac{1}{p_1}, 1\Big), \quad r_1 - r_2 > \Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+, \tag{4.181}$$

Figure 4.4, p. 198. The source spaces $S^{r_1}_{p_1q_1}\mathfrak{B}(\mathbb{Q}^2)$ are the same as in Theorem 4.15. Recall that $f \in S^{r_1}_{p_1q_1}\mathfrak{B}(\mathbb{Q}^2)$ can be uniquely represented as

$$f = \sum_{k\in\mathbb{N}^2_{-1}} \sum_{m\in\mathbb{P}^F_k} d^2_{km}(f)\, v_{km} \tag{4.182}$$

where

$$\|f\,|S^{r_1}_{p_1q_1}\mathfrak{B}(\mathbb{Q}^2)\| \sim \Big(\sum_{k\in\mathbb{N}^2_{-1}} 2^{(k_1+k_2)(r_1-\frac{1}{p_1})q_1} \Big(\sum_{m\in\mathbb{P}^F_k} |d^2_{km}(f)|^{p_1}\Big)^{q_1/p_1}\Big)^{1/q_1} \tag{4.183}$$

with the usual modifications if $p_1=\infty$ and/or $q_1=\infty$. This coincides with (4.123)–(4.125), where

$$S^{r_1}_{p_1q_1}\mathfrak{B}(\mathbb{Q}^2) = S^{r_1}_{p_1q_1}B(\mathbb{Q}^2), \tag{4.184}$$

complemented by (4.126), (4.127). As there $\{v_{km}\}$ is the Faber system (4.116)–(4.118) and $d^2_{km}(f)$ are the mixed differences (4.119)–(4.122). As for the spaces $S^{r_2}_{p_2q_2}\mathfrak{B}(\mathbb{Q}^2)$ we refer to Section 3.3, where one finds definitions, comments, and properties on which we rely in what follows.

Theorem 4.20. (i) *Let* $0<p_1,p_2,q_1,q_2\le\infty$ *and*

$$\frac{1}{p_1} < r_1 < 1+\min\Big(\frac{1}{p_1},1\Big), \quad 0<r_2<r_1-\Big(\frac{1}{p_1}-\frac{1}{p_2}\Big)_+, \tag{4.185}$$

Figure 4.4, p. 198. Then the embedding

$$\mathrm{id}\colon\ S^{r_1}_{p_1q_1}\mathfrak{B}(\mathbb{Q}^2) \hookrightarrow S^{r_2}_{p_2q_2}\mathfrak{B}(\mathbb{Q}^2) \tag{4.186}$$

is compact. Furthermore,

$$g^{\mathrm{lin}}_l(\mathrm{id}) \le c\,\Big(\frac{l}{\log l}\Big)^{-r_1+r_2+(\frac{1}{p_1}-\frac{1}{p_2})_+} (\log l)^{(\frac{1}{q_2}-\frac{1}{q_1})_+} \tag{4.187}$$

for some $c>0$ *and all* $l\in\mathbb{N}$, $l\ge 2$.

(ii) *Let* $1\le p\le\infty$ *and*

$$\frac{1}{p} < r_2 < r_1 < 1+\frac{1}{p}. \tag{4.188}$$

Then the embedding

$$\mathrm{id}\colon\ S^{r_1}_{pp}B(\mathbb{Q}^2) \hookrightarrow S^{r_2}_{pp}B(\mathbb{Q}^2) \tag{4.189}$$

is compact. Furthermore,

$$g^{\mathrm{lin}}_l(\mathrm{id}) \sim \Big(\frac{l}{\log l}\Big)^{-r_1+r_2}, \quad l\in\mathbb{N},\ l\ge 2. \tag{4.190}$$

(iii) *Let* $1 \le p < \infty$ *and*

$$0 < r_2 < \frac{1}{p} < r_1 < 1 + \frac{1}{p}. \tag{4.191}$$

Then the embedding

$$\mathrm{id}\colon S^{r_1}_{p,1}\mathfrak{B}(\mathbb{Q}^2) \hookrightarrow S^{r_2}_{p,1}\mathfrak{B}(\mathbb{Q}^2) \tag{4.192}$$

is compact. Furthermore,

$$c_1\, l^{-r_1+r_2} \le g_l(\mathrm{id}) \le g_l^{\mathrm{lin}}(\mathrm{id}) \le c_2 \left(\frac{l}{\log l}\right)^{-r_1+r_2} \tag{4.193}$$

for some $c_1 > 0$, $c_2 > 0$ *and all* $l \in \mathbb{N}$, $l \ge 2$.

Proof. Step 1. We prove part (i) and abbreviate temporarily $\mathbb{Q}^2 = Q$. Let $f \in S^{r_1}_{p_1 q_1}\mathfrak{B}(Q)$ be given by (4.182), (4.183). We split f as in Step 1 of the proof of Theorem 4.15. In particular,

$$R^K f = \sum_{k_1+k_2>K} \sum_{m\in\mathbb{P}^F_k} d^2_{km}(f)\, v_{km}, \quad K \in \mathbb{N}, \tag{4.194}$$

is the remainder term. According to Definition 3.24 one has

$$\begin{aligned} &\|R^K f\,|S^{r_2}_{p_2 q_2}\mathfrak{B}(Q)\| \\ &\quad\le c\Big(\sum_{k_1+k_2>K} 2^{(k_1+k_2)(r_2-\frac{1}{p_2})q_2}\Big(\sum_{m\in\mathbb{P}^F_k} |d^2_{km}(f)|^{p_2}\Big)^{q_2/p_2}\Big)^{1/q_2}. \end{aligned} \tag{4.195}$$

Let $p_2 \ge p_1$. Then

$$\begin{aligned} \|R^K f\,|S^{r_2}_{p_2 q_q}\mathfrak{B}(Q)\| \le c\Big[&\sum_{k_1+k_2>K} 2^{-(k_1+k_2)(r_1-r_2-\frac{1}{p_1}+\frac{1}{p_2})q_2}\, 2^{(k_1+k_2)(r_1-\frac{1}{p_1})q_2} \\ &\times \Big(\sum_{m\in\mathbb{P}^F_k} |d^2_{km}(f)|^{p_1}\Big)^{q_2/p_1}\Big]^{1/q_2}. \end{aligned} \tag{4.196}$$

If, in addition, $q_2 \ge q_1$, then one obtains that

$$\|R^K f\,|S^{r_2}_{p_2 q_2}\mathfrak{B}(Q)\| \le c\, 2^{-K(r_1-r_2-\frac{1}{p_1}+\frac{1}{p_2})} \|f\,|S^{r_1}_{p_1 q_1}\mathfrak{B}(Q)\| \tag{4.197}$$

where we used (4.183). Let again $p_2 \ge p_1$ but now $q_2 < q_1$. Then it follows from (4.196) that

$$\begin{aligned} \|R^K f\,|S^{r_2}_{p_2 q_2}\mathfrak{B}(Q)\| \le c\Big(&\sum_{k_1+k_2>K} 2^{(k_1+k_2)(r_1-\frac{1}{p_1})q_1}\Big(\sum_{m\in\mathbb{P}^F_k} |d^2_{km}(f)|^{p_1}\Big)^{q_1/p_1}\Big)^{1/q_1} \\ &\times \Big(\sum_{k_1+k_2>K} 2^{-(k_1+k_2)(r_1-\frac{1}{p_1}-r_2+\frac{1}{p_2})\frac{q_1 q_2}{q_1-q_2}}\Big)^{\frac{1}{q_2}-\frac{1}{q_1}} \end{aligned} \tag{4.198}$$

with $\varepsilon = r_1 - \frac{1}{p_1} - r_2 + \frac{1}{p_2} > 0$. The second factor can be estimated from above by

$$\Big(\sum_{j=K}^{\infty} (j-K+K)\, 2^{-\varepsilon j \frac{q_1 q_2}{q_1 - q_2}}\Big)^{\frac{1}{q_2}-\frac{1}{q_1}} \le cK^{\frac{1}{q_2}-\frac{1}{q_1}} 2^{-K\varepsilon} + 2^{-K\varepsilon}\Big(\sum_{l=1}^{\infty} l\, 2^{-\varepsilon l}\Big)^{\frac{1}{q_2}-\frac{1}{q_1}} \le c'\, K^{\frac{1}{q_2}-\frac{1}{q_1}}\, 2^{-K\varepsilon}. \tag{4.199}$$

In other words, if $p_2 \ge p_1$ then

$$\|R^K f \,|S^{r_2}_{p_2 q_2}\mathfrak{B}(Q)\| \le c\, K^{(\frac{1}{q_2}-\frac{1}{q_1})_+}\, 2^{-K(r_1 - r_2 - \frac{1}{p_1} + \frac{1}{p_2})}\, \|f\,|S^{r_1}_{p_1 q_1}\mathfrak{B}(Q)\|. \tag{4.200}$$

Let $p_2 < p_1$ and, hence, $r_2 < r_1$. We insert

$$\Big(\sum_{m\in\mathbb{P}^F_k} |d^2_{km}(f)|^{p_2}\Big)^{1/p_2} \le \Big(\sum_{m\in\mathbb{P}^F_k} |d^2_{km}(f)|^{p_1}\Big)^{1/p_1} 2^{(k_1+k_2)(\frac{1}{p_2}-\frac{1}{p_1})} \tag{4.201}$$

in (4.195). Afterwards one can argue as above now with $\varepsilon = r_1 - r_2 > 0$. Altogether one has

$$\|R^K f \,|S^{r_2}_{p_2 q_2}\mathfrak{B}(Q)\| \le c\, K^{(\frac{1}{q_2}-\frac{1}{q_1})_+}\, 2^{-K(r_1 - r_2 - (\frac{1}{p_1} - \frac{1}{p_2})_+)}\, \|f\,|S^{r_1}_{p_1 q_1}\mathfrak{B}(Q)\|. \tag{4.202}$$

Now one obtains (4.187) in the same way as in the proof of Theorem 4.15.

Step 2. We prove part (ii). By (4.123)–(4.125) and (4.184) we have

$$S^{r_1}_{pp}B(Q) = S^{r_1}_{pp}\mathfrak{B}(Q) \quad \text{and} \quad S^{r_2}_{pp}B(Q) = S^{r_2}_{pp}\mathfrak{B}(Q). \tag{4.203}$$

The estimate from above is covered by (4.187). By Definition 3.24 and Theorem 3.26 (ii),

$$D_1 \colon f \in S^{r_1}_{pp}B(Q) \hookrightarrow \big\{d^2_{km}(f) : k \in \mathbb{N}^2_{-1},\ m \in \mathbb{P}^F_k\big\} \tag{4.204}$$

is an isomorphic map of $S^{r_1}_{pp}B(Q)$ *onto* $s^{r_1}_{pp}b(Q)$. Here $s^{r_1}_{pp}b(Q)$ consists of all sequences

$$\mu = \big\{\mu_{km} \in \mathbb{C} : k \in \mathbb{N}^2_{-1},\ m \in \mathbb{P}^F_k\big\} \tag{4.205}$$

with

$$\|\mu\,|s^{r_1}_{pp}b(Q)\| = \Big(\sum_{k,m} 2^{(k_1+k_2)(r_1 - \frac{1}{p})p} |\mu_{km}|^p\Big)^{1/p} < \infty, \tag{4.206}$$

(usual modification if $p = \infty$). This is a special case of (4.182), (4.183). Similarly for D_2 with r_2 in (4.204). Let $a_l(\mathrm{id}_b)$ be the approximation numbers of the embedding

$$\mathrm{id}_b \colon s^{r_1}_{pp}b(Q) \hookrightarrow s^{r_2}_{pp}b(Q). \tag{4.207}$$

Let ℓ_p, $1 \le p \le \infty$,

$$\xi = \{\xi_j\}_{j=1}^{\infty} \in \ell_p, \quad \|\xi\,|\ell_p\| = \Big(\sum_{j=1}^{\infty} |\xi_j|^p\Big)^{1/p}, \tag{4.208}$$

be the usual complex sequence space. Let

$$d_1 \ge d_2 \ge \cdots \ge d_j \ge \cdots \to 0, \quad j \to \infty, \tag{4.209}$$

be a monotonically decreasing (= not increasing) sequence of positive numbers. Then

$$a_l(D) = d_l, \quad l \in \mathbb{N}, \quad D\xi = \{d_j \xi_j\}, \quad \{\xi_j\} \in \ell_p, \tag{4.210}$$

are the approximation numbers of the compact embedding D in ℓ_p, [Pie87, Proposition 2.9.5, p. 108], [CaS90, pp. 45/46]. Applied to the above situation one has

$$\{d_l\} \sim \{2^{-(k_1+k_2)(r_1-r_2)} : k \in \mathbb{N}^2_{-1}, \ m \in \mathbb{P}^F_k\}. \tag{4.211}$$

Recall that card $\mathbb{P}^F_k = 2^{k_1+k_2}$. One obtains by (4.145) that

$$d_l \sim \Big(\frac{l}{\log l}\Big)^{-r_1+r_2}, \quad l \in \mathbb{N}, \quad l \ge 2. \tag{4.212}$$

Then the desired estimate from below follows from

$$g_l^{\mathrm{lin}}(\mathrm{id}) \ge a_l(\mathrm{id}) \sim d_l \sim \Big(\frac{l}{\log l}\Big)^{-r_1+r_2}. \tag{4.213}$$

Step 3. We prove part (iii). The estimate from above is covered by (4.187). By (3.218) we have

$$\mathrm{id}_2 : S^{r_2}_{p,1}\mathfrak{B}(Q) \hookrightarrow L_{p_2}(Q), \quad r_2 - \frac{1}{p} = -\frac{1}{p_2}. \tag{4.214}$$

With id as in (4.192) and

$$\mathrm{id}_2 \circ \mathrm{id} = \mathrm{id}_1 : S^{r_1}_{p,1}\mathfrak{B}(Q) \hookrightarrow L_{p_2}(Q) \tag{4.215}$$

one obtains by (4.132) that

$$c\, l^{-r_1+\frac{1}{p}-\frac{1}{p_2}} \le g_l(\mathrm{id}_1) \le g_l(\mathrm{id}), \quad l \in \mathbb{N}. \tag{4.216}$$

The left-hand side of (4.193) follows now from $r_2 - \frac{1}{p} = -\frac{1}{p_2}$ in (4.214) and (4.216). □

Remark 4.21. In part (ii) we have not only an estimate but an equivalence. In addition, the spaces $S^{r_1}_{pp}B(\mathbb{Q}^2)$ and $S^{r_2}_{pp}B(\mathbb{Q}^2)$ can be characterised in terms of differences. We refer to Theorem 1.67 (i). One may choose $M = 2$ in (1.294), (1.299). If $\frac{1}{p} < r_2 < r_1 < 1$, then first differences, $M = 1$, are sufficient. We describe an example. Let

$$S^r\mathcal{C}(\mathbb{Q}^2) = S^r_{\infty\infty}B(\mathbb{Q}^2), \quad r > 0, \tag{4.217}$$

be the restriction of the *Hölder–Zygmund spaces with dominating mixed smoothness* $S^r\mathcal{C}(\mathbb{R}^2)$ according to (1.162)–(1.165) to $\mathbb{Q}^2$. According to Theorem 1.67 the spaces

$S^r\mathcal{C}(\mathbb{Q}^2)$ with $0 < r < 1$ can be equivalently normed by

$$\begin{aligned} &\| f \,|S^r\mathcal{C}(\mathbb{Q}^2)\| \\ &\sim \sup_{\substack{x=(x_1,x_2)\in Q \\ y=(y_1,y_2)\in Q}} \Big[|f(x)| + \frac{|f(x_1,x_2)-f(y_1,x_2)|}{|x_1-y_1|^r} \\ &\qquad + \frac{|f(x_1,x_2)-f(x_1,y_2)|}{|x_2-y_2|^r} \\ &\qquad + \frac{|f(x_1,x_2)-f(y_1,x_2)-f(x_1,y_2)+f(y_1,y_2)|}{|x_1-y_1|^r|x_2-y_2|^r} \Big]. \end{aligned} \tag{4.218}$$

Then it follows from part (ii) of the above theorem that

$$g_l^{\mathrm{lin}}\big(\mathrm{id}\colon S^{r_1}\mathcal{C}(\mathbb{Q}^2) \hookrightarrow S^{r_2}\mathcal{C}(\mathbb{Q}^2)\big) \sim \Big(\frac{l}{\log l}\Big)^{-r_1+r_2}, \quad 0 < r_2 < r_1 < 1, \tag{4.219}$$

$l \in \mathbb{N}$, $l \geq 2$, with $S^{r_1}\mathcal{C}(\mathbb{Q}^2)$, $S^{r_2}\mathcal{C}(\mathbb{Q}^2)$ normed by (4.218). Let $I = (0,1)$ be the unit interval in $\mathbb{R}$ and $\mathcal{C}^r(I) = B^r_{\infty\infty}(I)$. Then one has by Theorem 4.13 (i) that

$$g_l^{\mathrm{lin}}\big(\mathrm{id}\colon \mathcal{C}^{r_1}(I) \hookrightarrow \mathcal{C}^{r_2}(I)\big) \sim l^{-r_1+r_2}, \quad l \in \mathbb{N}. \tag{4.220}$$

In other words if one steps from spaces on I of order r to corresponding spaces with dominating mixed smoothness of the same order r in two or higher dimensions then one can expect that the main order of approximation, here expressed by (4.220), is preserved, but perturbed by powers of $\log l$, as here in (4.219). Quite obviously by the mentality of mathematicians such an observation opens a fierce battle about the correct power of the log-perturbation (if there is any). This will be also one of the main points of this book in what follows.

4.4 Sampling in logarithmic spaces with dominating mixed smoothness

4.4.1 Introduction and motivation

This Section 4.4 might be considered as the direct continuation of the preceding Section 4.3. In particular $\mathbb{Q}^2$ is again the unit square (4.113) in $\mathbb{R}^2$. In Theorem 4.15 we dealt with sampling numbers of the embedding (4.131). In particular if

$$\mathrm{id}\colon S^r_{pp}B(\mathbb{Q}^2) \hookrightarrow L_p(\mathbb{Q}^2) \tag{4.221}$$

with

$$1 \leq p \leq \infty, \quad \frac{1}{p} < r < 1 + \frac{1}{p}, \tag{4.222}$$

then one has

$$c_1\, l^{-r} (\log l)^{1-\frac{1}{p}} \le g_l(\mathrm{id}) \le g_l^{\mathrm{lin}}(\mathrm{id}) \le c_2\, l^{-r} (\log l)^{1-\frac{1}{p}+r} \tag{4.223}$$

for some $c_1 > 0$, $c_2 > 0$ and all $l \in \mathbb{N}$, $l \ge 2$. Here we used again that

$$S^r_{pp} B(\mathbb{Q}^2) = S^r_{pp} \mathfrak{B}(\mathbb{Q}^2), \quad 1 \le p \le \infty, \tag{4.224}$$

where $S^r_{pp} B(\mathbb{Q}^2)$ is the restriction of $S^r_{pp} B(\mathbb{R}^2)$ to $\mathbb{Q}^2$, (4.123)–(4.127). Recall that we have for the embedding (4.189) even the somewhat surprising equivalence (4.190) with (4.219) as a special case. But we wish to compare (4.223) with corresponding assertions for spaces on the unit interval $I = (0,1)$. Let

$$\mathrm{id}^I : B^r_{pp}(I) \hookrightarrow L_p(I), \quad 1 \le p \le \infty,\ r > 1/p. \tag{4.225}$$

Then it follows from (4.19) (or Theorem 4.11 if $r < 1 + \frac{1}{p}$) that

$$g_l(\mathrm{id}^I) \sim g_l^{\mathrm{lin}}(\mathrm{id}^I) \sim l^{-r}, \quad l \in \mathbb{N}. \tag{4.226}$$

The dominating factor l^{-r} in (4.223) is the same as in (4.226). But one has now indispensable powers of $\log l$. One may ask whether one can remove the log-factors on the right-hand side of (4.223) paying as less as possible. This can be done modifying the spaces $S^r_{pp} B(\mathbb{Q}^2)$ by their logarithmic generalisations $S^{r,b}_{pp} B(\mathbb{Q}^2)$.

We recall a few basic assertions about logarithmic spaces with dominating mixed smoothness. Let $S^{r,b}_{pq} B(\mathbb{R}^2)$ be the logarithmic spaces as introduced in Definition 1.79 and let $S^{r,b}_{pq} B(\mathbb{Q}^2)$ be its restriction to $\mathbb{Q}^2$ according to Definition 1.81 with $\Omega = \mathbb{Q}^2$. If $1 \le p = q \le \infty$ and $0 < r < M \in \mathbb{N}$ then one can expect that (1.369) with $Q = \mathbb{Q}^2$ are equivalent norms (although we did not check any detail as mentioned there). Of interest for us are extensions of the Faber representations

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(f)\, v_{km} \tag{4.227}$$

in (4.124) with the Faber basis $\{v_{km}\}$ as in (4.115)–(4.118) and the mixed differences $d^2_{km}(f)$ as in (4.119)–(4.122) from $S^r_{pq} B(\mathbb{Q}^2)$ to $S^{r,b}_{pq} B(\mathbb{Q}^2)$. This has been done in Theorem 3.35 for

$$0 < p,q < \infty, \quad \frac{1}{p} < r < 1 + \min\Big(\frac{1}{p}, 1\Big) \tag{4.228}$$

with

$$\| f \,| S^{r,b}_{pq} B(\mathbb{Q}^2) \| \sim \Big(\sum_{k \in \mathbb{N}^2_{-1}} 2^{(k_1+k_2)(r-\frac{1}{p})q} (2+k_1)^{bq} (2+k_2)^{bq} \Big(\sum_{m \in \mathbb{P}^F_k} |d^2_{km}(f)|^p \Big)^{q/p} \Big)^{1/q} \tag{4.229}$$

as the counterpart of (4.125). We commented in Remark 3.36 on this extension of Theorem 3.16 from $S^r_{pq}B(\mathbb{Q}^2)$ to $S^{r,b}_{pq}B(\mathbb{Q}^2)$. Furthermore we wish to incorporate some limiting cases. This can be done in the same way as in (4.126), (4.127).

Definition 4.22. Let

$$0 < p, q \le \infty, \quad \frac{1}{p} < r < 1 + \min\left(\frac{1}{p}, 1\right), \quad b \in \mathbb{R}. \tag{4.230}$$

Then $S^{r,b}_{pq}\mathfrak{B}(\mathbb{Q}^2)$ is the collection of all $f \in C(\mathbb{Q}^2)$ (or likewise $f \in L_1(\mathbb{Q}^2)$ or $f \in \mathbf{M}(\mathbb{Q}^2)$) which can be represented by (4.227) with

$$\|f \,|S^{r,b}_{pq}\mathfrak{B}(\mathbb{Q}^2)\| = \Big(\sum_{k \in \mathbb{N}^2_{-1}} 2^{(k_1+k_2)(r-\frac{1}{p})q} (2+k_1)^{bq} (2+k_2)^{bq} \Big(\sum_{m \in \mathbb{P}^F_k} |d^2_{km}(f)|^p \Big)^{q/p} \Big)^{1/q} < \infty \tag{4.231}$$

(usual modifications if $p = \infty$ and/or $q = \infty$).

Remark 4.23. This is the counterpart of (4.127) with (4.126). As there we refer for details to Sections 3.3.1, 3.3.2, in particular Theorem 3.26. As said above one may assume that

$$S^{r,b}_{pq}\mathfrak{B}(\mathbb{Q}^2) = S^{r,b}_{pq}B(\mathbb{Q}^2) \tag{4.232}$$

with p, q, r as in (4.228) is the restriction of $S^{r,b}_{pq}B(\mathbb{R}^2)$ to $\mathbb{Q}^2$. But in what follows we do not need this assertion. We rely exclusively on (4.227) with (4.231).

4.4.2 Basic assertions

Our motivation to deal with sampling numbers for logarithmic spaces comes from the comparison of (4.221)–(4.223) with the one-dimensional counterpart (4.225), (4.226). This may explain that we restrict ourselves to $p = q$, r as in (4.222). But there is little doubt that further assertions from Theorem 4.15 can be extended to the logarithmic case.

Theorem 4.24. *Let $\mathbb{Q}^2$ be the unit square (4.113) in $\mathbb{R}^2$. Let*

$$1 \le p \le \infty, \quad \frac{1}{p} < r < 1 + \frac{1}{p}, \quad b \ge 0. \tag{4.233}$$

Then the embedding

$$\mathrm{id}\colon S^{r,b}_{pp}\mathfrak{B}(\mathbb{Q}^2) \hookrightarrow L_p(\mathbb{Q}^2) \tag{4.234}$$

is compact. Furthermore,

$$c_1\, l^{-r} (\log l)^{-2b-\frac{1}{p}+1} \le g_l(\mathrm{id}) \le g^{\mathrm{lin}}_l(\mathrm{id}) \le c_2\, l^{-r} (\log l)^{r-b-\frac{1}{p}+1} \tag{4.235}$$

for some $c_1 > 0$, $c_2 > 0$ and all $l \in \mathbb{N}$, $l \ge 2$.

Proof. Step 1. We modify Step 1 of the proof of Theorem 4.15. Let $R^K f$ be as in (4.135). Put again for simplicity $Q = \mathbb{Q}^2$. Modification of (4.140) gives

$$\|R^K f \,|L_p(Q)\| \le c\, 2^{-Kr} K^{-b} \sum_{k_1+k_2>K} 2^{-(k_1+k_2-K)r} \frac{K^b}{(2+k_1)^b(2+k_2)^b} \times 2^{(k_1+k_2)(r-\frac{1}{p})}\, (2+k_1)^b(2+k_2)^b \Big(\sum_{m\in \mathbb{P}^F_k} |d^2_{km}(f)|^p \Big)^{1/p}. \tag{4.236}$$

Then one obtains as in (4.141) with $q = p$ that

$$\|R^K f \,|L_p(Q)\| \le c\, 2^{-Kr} K^{-b+1-\frac{1}{p}}\, \|f\,|S^{r,b}_{pp}\mathfrak{B}(Q)\|, \tag{4.237}$$

where we used (4.231). With K as in (4.145) this results in

$$g^{\mathrm{lin}}_l(\mathrm{id}) \le c\, l^{-r}(\log l)^{r-b-\frac{1}{p}+1}, \quad l\in\mathbb{N},\ l\ge 2. \tag{4.238}$$

Step 2. As for the estimate from below we rely on f in (4.153). The counterpart of (4.154) is now given by

$$\|f\,|S^{r,b}_{pp}\mathfrak{B}(Q)\| \sim 2^{lr} l^{\frac{1}{p}+2b}, \quad \|f\,|L_p(Q)\| \sim l, \tag{4.239}$$

where the first equivalence follows from (4.231),

$$\|f\,|S^{r,b}_{pp}\mathfrak{B}(Q)\| \sim 2^{lr}\Big(\sum_{k_1+k_2=l+1} (2+k_1)^{bp}(2+k_2)^{bp}\Big)^{1/p} \sim 2^{lr}\, l^{2b+\frac{1}{p}}. \tag{4.240}$$

The left-hand side of (4.235) can be obtained now from (4.239) in the same way as in Step 4 of the proof of Theorem 4.15. □

Corollary 4.25. *Let*

$$1\le p\le\infty, \quad \frac{1}{p} < r < 1+\frac{1}{p}. \tag{4.241}$$

Then the embedding

$$\mathrm{id}\colon\ S^{r,r+1-\frac{1}{p}}_{pp}\mathfrak{B}(\mathbb{Q}^2) \hookrightarrow L_p(\mathbb{Q}^2) \tag{4.242}$$

is compact. Furthermore,

$$c_1\, l^{-r}(\log l)^{\frac{1}{p}-1-2r} \le g_l(\mathrm{id}) \le g^{\mathrm{lin}}_l(\mathrm{id}) \le c_2\, l^{-r} \tag{4.243}$$

for some $c_1>0$, $c_2>0$ *and all* $l\in\mathbb{N}$, ≥ 2.

Proof. This is a special case of Theorem 4.24. □

Remark 4.26. The interest in the special case considered in Corollary 4.25 comes from the motivation described in Section 4.4.1. According to (4.225), (4.226) smoothness r for spaces on the interval $I = (0,1)$ ensures linear sampling in a constructive way of order $\sim l^{-r}$. We ask for a counterpart in the context of spaces with dominating mixed smoothness in $\mathbb{Q}^2$. The above corollary gives a satisfactory answers, although not of the same final character as in case of the interval I. Let again

$$K_l = \log l - \log\log l + \varepsilon, \quad \text{where } 0 \le \varepsilon < 1, \ l \in \mathbb{N}, \ l \ge 2, \tag{4.244}$$

and

$$M^{K_l} f = \sum_{k_1+k_2 \le K_l} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(f)\, v_{km}, \tag{4.245}$$

be as in (4.176), (4.177). As mentioned there $M^{K_l} f$ evaluates $f \in C(\mathbb{Q}^2)$ in $\sim l$ points in $\mathbb{Q}^2$. We interpret again M^{K_l} as an operator of rank $M^{K_l} \sim l$. Then it follows from the above corollary that

$$\begin{aligned} &c_1\, l^{-r} (\log l)^{\frac{1}{p}-1-2r} \le g^{\text{lin}}_l(\text{id}) \\ &\le c_2 \big\| \text{id} - M^{K_l} : S^{r,r+1-\frac{1}{p}}_{pp} \mathfrak{B}(\mathbb{Q}^2) \hookrightarrow L_p(\mathbb{Q}^2) \big\| \le c_3\, l^{-r} \end{aligned} \tag{4.246}$$

for some $c_1 > 0$, $c_2 > 0$, $c_3 > 0$ and all $l \in \mathbb{N}$, $l \ge 2$. This is a

constructive algorithm which produces the desired approximation l^{-r}.

But it shows also that one has indispensably to pay a price by a logarithmic reinforcement of the same smoothness r as in case of spaces on the interval $I = (0,1)$. In addition there remains the gap $(\log l)^{2r+1-\frac{1}{p}}$ between both sides in (4.246). But it can be sealed treating the interior terms in (4.231) with $k \in \mathbb{N}^2_0$ and the boundary terms with $k \in \mathbb{N}^2_{-1} \setminus \mathbb{N}^2_0$ differently. This will be done in what follows.

4.4.3 The spaces $S^{r,(b)}_{pq} B(\mathbb{Q}^2)$, main assertions

We characterised in Theorem 3.35 the spaces $S^{r,b}_{pq} B(\mathbb{Q}^2)$ with $b \in \mathbb{R}$ and p, q, r as in (3.258) in terms of Faber bases. We incorporated in Definition 4.22 the limiting cases with $\max(p,q) = \infty$ and called the outcome $S^{r,b}_{pq} \mathfrak{B}(\mathbb{Q}^2)$ because it is not clear (or at least not covered by our arguments) whether also $S^{r,b}_{pq} \mathfrak{B}(\mathbb{Q}^2)$ with $\max(p,q) = \infty$ is the restriction of $S^{r,b}_{pq} B(\mathbb{R}^2)$ to $\mathbb{Q}^2$. But otherwise one has (4.232). Motivated by the preceding Section 4.4.2, in particular by Remark 4.26, we modify Definition 4.22 dealing differently with the interior terms $k \in \mathbb{N}^2_0$ in (4.231) and the boundary terms $k \in \mathbb{N}^2_{-1} \setminus \mathbb{N}^2_0$. We enlarge the spaces $S^{r,b}_{pq} \mathfrak{B}(\mathbb{Q}^2)$ such that the right-hand side of (4.243) is preserved but the left-hand side improves to l^{-r} without the log-term. The corresponding spaces are sandwiched between $S^{r}_{pq} \mathfrak{B}(\mathbb{Q}^2)$ and $S^{r,b}_{pq} \mathfrak{B}(\mathbb{Q}^2)$ and denoted by $S^{r,(b)}_{pq} B(\mathbb{Q}^2)$ (where one can use B instead of $\mathfrak{B}$, because they cannot be mixed with restrictions of corresponding spaces on $\mathbb{R}^2$). We rely again on the representation (4.227) of $f \in C(\mathbb{Q}^2)$ with the Faber basis $\{v_{km}\}$ as in (4.115)–(4.118) and the mixed differences $d^2_{km}(f)$ as in (4.119)–(4.122).

Definition 4.27. Let

$$0 < p, q \le \infty, \qquad \frac{1}{p} < r < 1 + \min\Big(\frac{1}{p}, 1\Big), \qquad b \in \mathbb{R}. \tag{4.247}$$

Then $S^{r,(b)}_{pq} B(\mathbb{Q}^2)$ is the collection of all $f \in C(\mathbb{Q}^2)$ (or likewise $f \in L_1(\mathbb{Q}^2)$ or $f \in \mathbf{M}(\mathbb{Q}^2)$) which can be represented by

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(f)\, v_{km} \tag{4.248}$$

with

$$\begin{aligned} &\big\| f \,| S^{r,(b)}_{pq} B(\mathbb{Q}^2) \big\| \\ &\quad = \bigg(\sum_{k \in \mathbb{N}^2_{-1}} 2^{(k_1+k_2)(r-\frac{1}{p})q} \big(1 + (1+k_1)(1+k_2) \big)^{bq} \Big(\sum_{m \in \mathbb{P}^F_k} |d^2_{km}(f)|^p \Big)^{q/p} \bigg)^{1/q} \\ &\quad < \infty \end{aligned} \tag{4.249}$$

(usual modification if $p = \infty$ and/or $q = \infty$).

Remark 4.28. We add a few comments. If f is given by (4.248), (4.249) with $\mu_{km} \in \mathbb{C}$ in place of $d^2_{km}(f)$ then it follows as before, proof of Theorems 3.13, 3.16, that the corresponding series converges unconditionally in $C(\mathbb{Q}^2)$ and that the representation (4.248) is unique with $\mu_{km} = d^2_{km}(f)$. The terms with $k \in \mathbb{N}^2_0$ in (4.249) and (4.231) are essentially the same (up to immaterial equivalence constants). The situation is different if $k \in \mathbb{N}^2_{-1} \setminus \mathbb{N}^2_0$ (hence either $k_1 = -1$ and/or $k_2 = -1$). These are boundary terms. Put again $Q = \mathbb{Q}^2$ for simplicity. Let $B^r_{pq}(\partial Q)$ be the same space as in (3.121) and let $\mathrm{tr}_{\partial Q}$ be the trace operator according to (3.122). Then

$$\mathrm{tr}_{\partial Q} : f \mapsto f|\partial Q \quad \text{with } \mathrm{tr}_{\partial Q} : S^{r,(b)}_{pq} B(Q) \hookrightarrow B^r_{pq}(\partial Q). \tag{4.250}$$

This follows from (4.248), (4.249) with $(1+k_1)(1+k_2) = 0$ and

$$\mathrm{tr}_{\partial Q} f = \big(f^{\partial Q} + f^{\partial^2 Q} \big)|\partial Q \tag{4.251}$$

where

$$f^{\partial Q}(x) = v_0(x_1) \sum_{k_2=0}^{\infty} \sum_{m_2=0}^{2^{k_2}-1} d^2_{(-1,k_2),(0,m_2)}(f)\, v_{k_2 m_2}(x_2) + ++ \tag{4.252}$$

and $f^{\partial^2 Q}(x)$ have the same meaning as in (3.98), (3.99), based on (4.117), (4.120). By Theorem 3.16 and its proof it follows that $\mathrm{tr}_{\partial Q}$ maps $S^{r,(b)}_{pq}B(Q)$ onto $B^r_{pq}(\partial Q)$ and that $\mathrm{ext}_{\partial Q}$ in (3.150),

$$\mathrm{ext}_{\partial Q}\colon B^r_{pq}(\partial Q) \hookrightarrow S^{r,(b)}_{pq}B(Q), \quad \mathrm{tr}_{\partial Q} \circ \mathrm{ext}_{\partial Q} = \mathrm{id}, \tag{4.253}$$

is a common extension operator in all admitted spaces $S^{r,(b)}_{pq}B(Q)$. Recall that $a_+ = \max(a,0)$ with $a \in \mathbb{R}$.

Theorem 4.29. *Let $\mathbb{Q}^2$ be the unit square (4.113) in $\mathbb{R}^2$. Let*

$$1 \le p \le \infty, \quad \frac{1}{p} < r < 1 + \frac{1}{p}, \quad b \ge 0. \tag{4.254}$$

Then the embedding

$$\mathrm{id}\colon S^{r,(b)}_{pp}B(\mathbb{Q}^2) \hookrightarrow L_p(\mathbb{Q}^2) \tag{4.255}$$

is compact. Furthermore,

$$c_1\, l^{-r} \le g_l(\mathrm{id}) \le g_l^{\mathrm{lin}}(\mathrm{id}) \le c_2\, l^{-r}\, (\log l)^{(r-b-\frac{1}{p}+1)_+} \tag{4.256}$$

for some $c_1 > 0$, $c_2 > 0$ and all $l \in \mathbb{N}$, $l \ge 2$.

Proof. Step 1. We follow the proof of Theorem 4.24 and indicate the necessary modifications. We split again the representation (4.248) into a main part as in (4.134) and a remainder part as in (4.135) where special attention must be paid to boundary terms,

$$M^{K,L} f = \sum_{\substack{k_1+k_2 \le K \\ k \in \mathbb{N}_0^2}} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(f)\, v_{km} + \sum_{\substack{k_1+k_2 \le L \\ k \in \mathbb{N}^2_{-1} \setminus \mathbb{N}^2_0}} \cdots \tag{4.257}$$

and

$$R^{K,L} f = \sum_{\substack{k_1+k_2 > K \\ k \in \mathbb{N}_0^2}} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(f)\, v_{km} + \sum_{\substack{k_1+k_2 > L \\ k \in \mathbb{N}^2_{-1} \setminus \mathbb{N}^2_0}} \cdots, \tag{4.258}$$

$K \in \mathbb{N}$, $L \in \mathbb{N}$. Here the second summand in (4.257) is of type (4.252) with $k_2 = 0, \ldots, L$ in place of $k_2 \in \mathbb{N}_0$. The first summand in (4.258) gives again the estimate (4.237). By the same arguments as in (4.236) (with $b = 0$ and summations of type $k_2 = 0, \ldots, L$) one can estimate the second summand in (4.258) as in (4.237) with 2^{-Lr} in place of $2^{-Kr} K^{-b+1-\frac{1}{p}}$. Choosing K as in (4.145) and $2^L \sim l$ one obtains

$$g_l^{\mathrm{lin}}(\mathrm{id}) \le c\, l^{-r}\, (\log l)^{(r-b-\frac{1}{p}+1)_+}, \quad l \in \mathbb{N}, \; l \ge 2. \tag{4.259}$$

Step 2. The left-hand side of (4.256) is essentially a one-dimensional affair. We rely again on (4.147), (4.148). Let Γ be a set of at most 2^l points in $\mathbb{Q}^2$. Let

$$Q_{m_j} = (0,1) \times \big(2^{-l-1} m_j, 2^{-l-1}(m_j+1)\big), \quad 0 \le m_1 < m_2 < \cdots < m_{2^l} \le 2^{l+1}-1 \tag{4.260}$$

be 2^l rectangles with $Q_{m_j} \cap \Gamma = \emptyset$. Let

$$f(x) = v_0(x_1) \sum_{j=1}^{2^l} v_{l+1,m_j}(x_2) \tag{4.261}$$

be the counterpart of (4.149) consisting of Faber functions according to (4.117) with $k_1 + 1 = 0$. Then

$$\| f \,|S^{r,(b)}_{pp} B(\mathbb{Q}^2)\| \sim 2^{lr}, \quad \| f \,|L_p(\mathbb{Q}^2)\| \sim 1, \tag{4.262}$$

is the counterpart of (4.150), (4.151). This proves the left-hand side of (4.256). □

Next we formulate the counterparts both of Corollary 4.25 and (4.246). Let

$$K_l = \log l - \log\log l + \varepsilon, \quad L_l = \log l + \delta, \quad 0 \le \varepsilon, \delta < 1, \; l \in \mathbb{N}, \; l \ge 2. \tag{4.263}$$

Then the linear operator M_l,

$$M_l f = \sum_{\substack{k_1+k_2 \le K_l \\ k \in \mathbb{N}_0^2}} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(f)\, v_{km} + \sum_{\substack{k_1+k_2 \le L_l \\ k \in \mathbb{N}^2_{-1} \setminus \mathbb{N}^2_0}} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(f)\, v_{km} \tag{4.264}$$

evaluates $f \in C(\mathbb{Q}^2)$ at $\sim l$ points in $\overline{\mathbb{Q}^2}$.

Corollary 4.30. *Let*

$$1 \le p \le \infty, \quad \frac{1}{p} < r < 1 + \frac{1}{p}. \tag{4.265}$$

Then the embedding

$$\mathrm{id}\colon S^{r,(r+1-\frac{1}{p})}_{pp} B(\mathbb{Q}^2) \hookrightarrow L_p(\mathbb{Q}^2) \tag{4.266}$$

is compact. Furthermore,

$$\big\| \mathrm{id} - M_l \colon S^{r,(r+1-\frac{1}{p})}_{pp} B(\mathbb{Q}^2) \hookrightarrow L_p(\mathbb{Q}^2) \big\| \sim g_l(\mathrm{id}) \sim g^{\mathrm{lin}}_l(\mathrm{id}) \sim l^{-r}, \tag{4.267}$$

$l \in \mathbb{N}$ (*universal order-optimal constructive algorithms*).

Proof. This follows from Theorem 4.29 and its proof. □

4.4.4 Higher dimensions: comments, problems, proposals

We described in Section 3.2.5 how to extend the theory for the spaces $S^r_{pq} B(\mathbb{Q}^2)$ and $S^1_p W(\mathbb{Q}^2)$ based on Faber expansions from two to higher dimensions. These comments provide a solid background to generalise the assertions about sampling numbers in Sections 4.3.2, 4.3.3 from two to higher dimensions. This is of interest for its own sake, but also to compare related results with the literature mentioned in

Section 4.3.2, dealing mostly with all dimensions $n \in \mathbb{N}$. However this will not be done in this book. The only exception later on will be the case $u = 1$ in the context of Theorem 4.15, Corollary 4.16, hence the target space $L_1(\mathbb{Q}^2)$, which is related to numerical integration. But we wish to add a few more detailed comments and proposals in connection with sampling numbers for logarithmic spaces with dominating mixed smoothness as considered in the preceding Sections 4.4.2, 4.4.3, always guided by the attempt to preserve the one-dimensional assertions (4.225), (4.226) in higher dimensions paying as less as possible. Let again

$$\mathbb{Q}^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : 0 < x_l < 1\} \tag{4.268}$$

be the unit cube in $\mathbb{R}^n$, $n \geq 2$. Let

$$\{v_0,\ v_1,\ v_{jm} : j \in \mathbb{N}_0;\ m = 0, \dots, 2^j - 1\} \tag{4.269}$$

be the Faber system (4.57)–(4.59) on the unit interval $I = (0,1)$ on the real line $\mathbb{R}$. This coincides with (3.1)–(3.3), Figure 2.1, p. 64. Recall that $\{v_{km}\}$ in (4.116)–(4.118) is the corresponding Faber system in $\mathbb{Q}^2$ originating from (4.269) by the usual (tensor)-product procedure. There is an obvious generalisation

$$\{v_{km} : k \in \mathbb{N}^n_{-1};\ m \in \mathbb{P}^{F,n}_k\}, \tag{4.270}$$

indicating now the dimension n. This is the same ass in (3.178), in particular $\mathbb{N}^n_{-1}$ is the collection of all $k = (k_1, \dots, k_n)$ with $k_j \in \mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$. Similarly one extends the mixed differences $d^2_{km}(f)$ in (4.119)–(4.122) from two to n dimensions, denoted as in Section 3.2.5 as $d^2_{km}(f)^n$. In generalisation of Theorem 3.10 the Faber system $\{v_{km}\}$ in (4.270) is a (conditional) basis in $C(\mathbb{Q}^n)$, hence

$$f = \sum_{k \in \mathbb{N}^n_{-1}} \sum_{m \in \mathbb{P}^{F,n}_k} d^2_{km}(f)^n \, v_{km}, \quad f \in C(\mathbb{Q}^n), \tag{4.271}$$

interpreted as in Theorem 3.10. In Section 3.2.5 we described some crucial properties of the spaces $S^r_{pq}B(\mathbb{Q}^n)$ and $S^1_pW(\mathbb{Q}^n)$ extending corresponding assertions from $n = 2$ to $n \geq 3$. This will be complemented now by several points, some of them might be considered as research proposals.

1. The spaces $S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)$ and $S^1_pW(\mathbb{Q}^n)$. Let

$$0 < p, q \leq \infty, \quad \frac{1}{p} < r < 1 + \min\Big(\frac{1}{p}, 1\Big). \tag{4.272}$$

First one may ask for Faber representations of the restrictions $S^r_{pq}B(\mathbb{Q}^n)$ of $S^r_{pq}B(\mathbb{R}^n)$ to $\mathbb{Q}^n$ generalising Theorems 3.13, 3.16. One can incorporate some limiting cases as in (4.127) resulting in the spaces $S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)$ with p, q, r as in (4.272). Taking these spaces and also the corresponding Sobolev spaces $S^1_pW(\mathbb{Q}^n)$, $1 < p < \infty$, as source spaces one can try to find the n-dimensional counterparts of Theorem 4.15 and

Corollary 4.16. Of interest is the dependence of the log-powers on $n \in \mathbb{N}$, also in comparison with what is known in the literature mentioned there (mostly dealing with corresponding spaces on the n-torus $\mathbb{T}^n$). In connection with numerical integration we return later on to the special target space $L_1(\mathbb{Q}^n)$.

2. The spaces $S^{r,b}_{pq}\mathfrak{B}(\mathbb{Q}^2)$ and $S^{r,(b)}_{pq}B(\mathbb{Q}^2)$. We are back to $n = 2$. In Definitions 4.22 and 4.27 we introduced the spaces $S^{r,b}_{pq}\mathfrak{B}(\mathbb{Q}^2)$ (assuming (4.232)) and $S^{r,(b)}_{pq}B(\mathbb{Q}^2)$ for all p, q, r in (4.272) and $b \in \mathbb{R}$ (modifying smoothness r by a log-term). But we restricted the considerations in Sections 4.4.2, 4.4.3 to the special embeddings (4.234), (4.255) reaching in Corollary 4.30 the satisfactory assertion (4.267). Compared with (4.130) in Theorem 4.15 we specified p, q, u by $1 \le p = q = u \le \infty$. But it might well be of interest to clarify the q-dependence as in Theorem 4.15 and to replace the distinguished target space $L_p(\mathbb{Q}^2)$ by $L_u(\mathbb{Q}^2)$, $0 < u \le \infty$. In other words, one may ask what can be said about the sampling numbers $g_l(\mathrm{id})$ and $g^{\mathrm{lin}}_l(\mathrm{id})$ if one generalises id in (4.234), (4.255) by

$$\mathrm{id}\colon S^{r,b}_{pq}\mathfrak{B}(\mathbb{Q}^2) \hookrightarrow L_u(\mathbb{Q}^2) \tag{4.273}$$

or

$$\mathrm{id}\colon S^{r,(b)}_{pq}B(\mathbb{Q}^2) \hookrightarrow L_u(\mathbb{Q}^2) \tag{4.274}$$

with p, q, r as in (4.272) and $0 < u \le \infty$.

3. The spaces $S^{r,b}_{pq}\mathfrak{B}(\mathbb{Q}^n)$. The generalisation of Definition 4.22 from $n = 2$ to $n \in \mathbb{N}$, $n \ge 3$, is straightforward. Then one can ask for counterparts of corresponding assertions in Section 4.4.2 and their possible extensions with id in (4.273).

4. The spaces $S^{r,(b)}_{pq}B(\mathbb{Q}^n)$. The situation is now different compared with the spaces $S^{r,b}_{pq}\mathfrak{B}(\mathbb{Q}^n)$ discussed above. First we stick again at $n = 2$. The motivation for the construction (4.249) is twofold. On the one hand we wish to have the trace assertions (4.250)–(4.253) ensuring the estimates from below in (4.256), (4.267). On the other hand $b > 0$ seems to be indispensable to obtain the same estimate from above. But it is not clear whether $b = r + 1 - \frac{1}{p}$ in Corollary 4.30 is the best possible (smallest) choice. One could also replace $(1+k_1)^b(1+k_2)^b$ in (4.249) by a (symmetric) function $h(k_1, k_2)$ asking for necessary and sufficient conditions ensuring (4.267). The question arises how to extend these considerations from $\mathbb{Q}^2$ to $\mathbb{Q}^n$ with $n \ge 3$. To provide a better understanding we modify first (4.249) replacing

$$\big(1+(1+k_1)(1+k_2)\big)^b \quad \text{by} \quad 1+\sum_{j=1}^{2}(1+k_j)^{b_1}+(1+k_1)^{b_1+b_2}(1+k_2)^{b_1+b_2} \tag{4.275}$$

with $0 \le b_1 < b_1 + b_2$. The corresponding space, denoted by $S^{r,(b_1,b_2)}_{pq}B(\mathbb{Q}^2)$, coincides with $S^{r,(b)}_{pq}B(\mathbb{Q}^2)$ if $b_1 = 0$, $b_2 = b$. There is a counterpart of the trace assertions (4.250)–(4.253), in particular

$$\mathrm{tr}_{\partial\mathbb{Q}^2}\colon S^{r,(b_1,b_2)}_{pq}B(\mathbb{Q}^2) \hookrightarrow B^{r,b_1}_{pq}(\partial\mathbb{Q}^2), \tag{4.276}$$

in modification of (3.121) with $B^{r,b_1}_{pq}(I_j)$ in place of $B^r_{pq}(I_j)$. Here $B^{r,b_1}_{pq}(I)$ are the logarithmic spaces on $I = (0,1)$ as considered in Remark 1.76. For $\max(p,q) < \infty$ one has Faber expansions as in Theorem 3.35 (ii). The counterpart of the benchmark (4.225), (4.226) is now given by

$$\mathrm{id}^I : B^{r,b_1}_{pp}(I) \hookrightarrow L_p(I), \quad 1 \le p \le \infty, \ \tfrac{1}{p} < r < 1 + \tfrac{1}{p}, \ b_1 \ge 0, \tag{4.277}$$

and

$$g_l(\mathrm{id}^I) \sim g^{\mathrm{lin}}_l(\mathrm{id}^I) \sim l^{-r}(\log l)^{-b_1}, \quad l \in \mathbb{N}, \ l \ge 2. \tag{4.278}$$

This can be proved by similar, but simpler arguments as in the proof of Theorem 4.24. Afterwards one can extend the assertions in Section 4.4.3 from $S^{r,(b)}_{pp}B(\mathbb{Q}^2)$ to $S^{r,(b_1,b_2)}_{pp}B(\mathbb{Q}^2)$. This might be not so interesting for its own sake but it paves the way for a better understanding of the following proposal asking for an n-dimensional counterpart of the spaces $S^{r,(b)}_{pq}B(\mathbb{Q}^2)$ in Definition 4.27 or of the spaces $S^{r,(b_1,b_2)}_{pq}B(\mathbb{Q}^2)$. Now one has to deal with all edges and faces of $\partial\mathbb{Q}^n$. Let $n \ge 2$. We mean by $\{j_1, \ldots, j_d\} \subset \{1, \ldots, n\}$ all subsets of $\{1, \ldots, n\}$ where $d = 1, \ldots, n$.

Let

$$0 < p, q \le \infty, \quad \frac{1}{p} < r < 1 + \min\Big(\frac{1}{p}, 1\Big), \quad b_l \ge 0; \ l = 1, \ldots, n. \tag{4.279}$$

Then $S^{r,(b_1,\ldots,b_n)}_{pq}B(\mathbb{Q}^n)$ *is the collection of all* $f \in C(\mathbb{Q}^n)$ (*or likewise* $f \in L_1(\mathbb{Q}^n)$ *or* $f \in \mathbf{M}(\mathbb{Q}^n)$) *which can be represented by*

$$f = \sum_{k \in \mathbb{N}^n_{-1}} \sum_{m \in \mathbb{P}^{F,n}_k} d^2_{km}(f)^n \, v_{km} \tag{4.280}$$

with

$$\begin{aligned} &\| f \,|S^{r,(b_1,\ldots,b_n)}_{pq}B(\mathbb{Q}^n)\| \\ &= \bigg(\sum_{k \in \mathbb{N}^n_{-1}} 2^{(k_1+\cdots+k_n)(r-\frac{1}{p})q} \Big[1 + \sum_{\{j_1,\ldots,j_d\} \subset \{1,\ldots,n\}} \prod_{a=1}^d (1+k_{j_a})^{b_1+\cdots+b_d} \Big]^q \\ &\quad \times \Big(\sum_{m \in \mathbb{P}^{F,n}_k} |d^2_{km}(f)^n|^p \Big)^{q/p} \bigg)^{1/q} < \infty, \end{aligned} \tag{4.281}$$

(*usual modification if* $p = \infty$ *and/or* $q = \infty$).

This might be a reasonable proposal to extend Definition 4.27 from $n = 2$ to $3 \le n \in \mathbb{N}$ always having in mind that one wishes to reach at the end assertions of type (4.267) or the slightly extended benchmark (4.277), (4.278). This may also justify the restriction $b_l \ge 0$ in (4.279). The 2^n corner points of $\mathbb{Q}^n$ are characterised by $k_1 = \cdots = k_n = -1$. Then [...] in (4.281) reduces to 1. For an l-dimensional face of $\mathbb{Q}^n$ with $l = 1, \ldots, n-1$ one has $k_j \in \mathbb{N}_0$ for l components of $k \in \mathbb{N}^n_{-1}$ and

$k_j = -1$ for the remaining $n-l$ components, say, $k_{l+1} = \cdots = k_n = -1$. Then $[\ldots]$ in (4.281) reduces to

$$[\ldots] = 1 + \sum_{\{j_1,\ldots,j_d\}\subset\{1,\ldots,l\}} \prod_{a=1}^{d} (1+k_{j_a})^{b_1+\cdots+b_d}. \tag{4.282}$$

In other words, climbing upwards the faces in $\partial\mathbb{Q}^n$ of $\mathbb{Q}^n$ by dimension, reaching finally $\mathbb{Q}^n$ itself, one adds additional terms in (4.282), enhancing the required logarithmic smoothness step by step. The goal to be reached is the counterpart of Corollary 4.30, hence

$$\big\| \mathrm{id} - M_l : S^{r,(b_1,\ldots,b_n)}_{pp} B(\mathbb{Q}^n) \hookrightarrow L_p(\mathbb{Q}^n) \big\| \sim g_l(\mathrm{id}) \sim g_l^{\mathrm{lin}}(\mathrm{id}) \sim l^{-r} \tag{4.283}$$

$l \in \mathbb{N}$, for suitably chosen $b_1 = 0$, $b_j > 0$, $j = 2,\ldots,n$. This has not yet been done and the above suggestions may be considered as a research proposal. In addition one may ask what happens if one replaces the target space $L_p(\mathbb{Q}^n)$ by $L_u(\mathbb{Q}^n)$, $0 < u \le \infty$, or the source space

$$S^{r,(b_1,\ldots,b_n)}_{pp} B(\mathbb{Q}^n) \quad \text{by} \quad S^{r,(b_1,\ldots,b_n)}_{pq} B(\mathbb{Q}^n).$$

5. Tractability. All calculations in Sections 4.4.2, 4.4.3 are explicit, based on Faber expansions. The constants in the inequalities and equivalences in Theorems 4.24, 4.29 and Corollaries 4.25, 4.30 (and also in the other sections of this Chapter 4) can be calculated or estimated. This may be not so interesting in the two-dimensional case treated there. However the situation is different if one deals with higher dimensions as suggested above. Then it is of interest how the constants depend on the dimension $n \in \mathbb{N}$. This is now a fashionable topic in the context of complexity theory, especially in connection with numerical integration. One may consult [NSTW09] which is a readable introduction to this field of research with historical comments and related references. It is also one of the major topics of [NoW08], [NoW09], studied there in detail. We borrow one idea from these sources (and the underlying literature), used there preferable in connection with numerical integration. To have a better control about the dependence of constants on the dimension $n \in \mathbb{N}$ and to force the problem to be tractable one introduces additional weights, scaling factors, either to variables (ordered by *importance*) or to dimensions. Let $\gamma = \{\gamma_d\}_{d=1}^{\infty}$ be a sequence of positive numbers, typically with $1 \ge \gamma_1 \ge \gamma_2 \ge \cdots \to 0$ if $d \to \infty$. Then

$$\begin{aligned} &\big\| f \,|S^{r,(b_1,\ldots,b_n)}_{pq,\gamma} B(\mathbb{Q}^n) \big\| \\ &= \Bigg(\sum_{k\in\mathbb{N}^n_{-1}} 2^{(k_1+\cdots+k_n)(r-\frac{1}{p})q} \Big[1 + \sum_{\{j_1,\ldots,j_d\}\subset\{1,\ldots,n\}} \tfrac{1}{\gamma_1\cdots\gamma_d} \prod_{a=1}^{d} (1+k_{j_a})^{b_1+\cdots+b_d} \Big]^q \\ &\qquad \times \Big(\sum_{m\in\mathbb{P}^{F,n}_k} |d^2_{km}(f)^n|^p \Big)^{q/p} \Bigg)^{1/q} < \infty \end{aligned} \tag{4.284}$$

(usual modification it $p = \infty$ and/or $q = \infty$) is a scaled dimension-sensitive version of (4.281). Climbing upwards the faces $\partial \mathbb{Q}^n$ by dimension there is now a double penalty or reinforcement:

> *Additional logarithmic smoothness, expressed by $b_l > 0$, ensuring that (4.278) can be extended to higher dimensions with equivalence constants which are independent of $l \in \mathbb{N}$, $l \geq 2$, and additional factors $\gamma_l > 0$ ensuring (**at the best**) that these equivalence constants are independent of $l \in \mathbb{N}$, $l \geq 2$, and of $n \in \mathbb{N}$.*

So far we introduced the spaces $S^{r,(b)}_{pq} B(\mathbb{Q}^2)$ in Definition 4.27 and their n-dimensional generalisations in (4.281), (4.284) in terms of the discrete differences (4.119)–(4.122) and their n-dimensional generalisations. One may ask whether one can replace these discrete quasi-norms by corresponding continuous quasi-norms. We refer for related quasi-norms to Remark 1.76 and (1.369). It would be desirable to control the constants, in particular their dependence on the dimension $n \in \mathbb{N}$, when switching from the above discrete quasi-norms to corresponding continuous quasi-norms of the indicated type. In [Tri10] we used the characterisations of $B^s_{pp}(\mathbb{Q}^n)$, $1 < p < \infty$, $0 < s < 1/p$, in terms of Haar wavelet bases according to Theorem 2.26 to compare related discrete norms with the usual continuous standard norms. In particular we controlled the dependence of corresponding constants on the dimension $n \in \mathbb{N}$. Maybe one can employ this technique also for Haar tensor bases and related spaces with dominating mixed smoothness. This can be transferred to Faber bases using the isomorphic (and even isometric) mapping in (3.194). This may pave the way to replace in (4.283) discrete norms of type (4.281), (4.284) by more handsome continuous norms without losing the control of the dependence of equivalence constants on $n \in \mathbb{N}$. Nothing has been done so far in these directions.

Chapter 5
Numerical integration

5.1 Preliminaries, integration in domains

5.1.1 Introduction, definitions

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Recall that *domain* means open set and that $C(\Omega)$ is the collection of all complex-valued continuous functions in $\overline{\Omega}$, furnished in the usual way with a norm. We introduced in Definition 4.1 the sampling numbers $g_k(\mathrm{id})$ and $g_k^{\mathrm{lin}}(\mathrm{id})$ for compact embeddings

$$\mathrm{id}\colon G_1(\Omega) \hookrightarrow G_2(\Omega), \quad \big\{G_1(\Omega), G_2(\Omega)\big\} \subset D'(\Omega). \tag{5.1}$$

Of special interest is now the target space $G_2(\Omega) = L_1(\Omega)$ whereas we assume that $G(\Omega) = G_1(\Omega)$ satisfies (4.2),

$$\mathrm{id}\colon G(\Omega) \hookrightarrow C(\Omega), \tag{5.2}$$

in the interpretation given there. This is near to *numerical integration* which means the attempt to recover the integral

$$\int_\Omega f(x)\,\mathrm{d}x, \quad f \in G(\Omega), \tag{5.3}$$

optimally from finitely many function values $f(x^j)$, $j = 1, \ldots, k$. Numerical integration is one of the major topics in analysis since ages. The classical and modern theory covers numerous books and thousands of papers. We refer in particular to the detailed and comprehensive study of multivariate integration and discrepancy from the point of view of tractability in [NoW09], based on [NoW08]. As an introduction to this flourishing field of research one may also consult [NSTW09]. The close connection of numerical integration with discrepancy will be the subject of the next Chapter 6. In this Chapter 5 we deal in Sections 5.2-5.4 with numerical integration based on Faber expansions and sampling as developed in the preceding chapters. We outlined this new approach in [Tri09]. This is a very specific point of view. But we wish to illustrate the close connection between numerical integration and some distinguished properties of function spaces.

Definition 5.1. Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $G(\Omega)$ be a quasi-Banach space satisfying (5.2). Let $k \in \mathbb{N}$.

(i) Then

$$g_k^{\text{lin}}\big(G(\Omega)\big) = \inf\Big[\sup\Big\{\int_\Omega \Big|f(x) - \sum_{l=1}^k f(x^l)\,h_l(x)\Big|\,\mathrm{d}x \,:\, f \in G(\Omega),\ \|f\,|G(\Omega)\| \le 1\Big\}\Big] \tag{5.4}$$

where the infimum is taken over all $\{x^l\}_{l=1}^k \subset \Omega$ and all $\{h_l\}_{l=1}^k \subset L_1(\Omega)$.

(ii) Then

$$\mathrm{Int}_k\big(G(\Omega)\big) = \inf\Big[\sup\Big\{\Big|\int_\Omega f(x)\,\mathrm{d}x - \sum_{l=1}^k a_l\, f(x^l)\Big| \,:\, f \in G(\Omega),\ \|f\,|G(\Omega)\| \le 1\Big\}\Big] \tag{5.5}$$

is the k-th *integral number* where the infimum is taken over all $\{x^l\}_{l=1}^k \subset \Omega$ and all $\{a_l\}_{l=1}^k \subset \mathbb{C}$.

Remark 5.2. Recall that $G(\Omega)$ is considered as a subspace of $D'(\Omega)$. Then (5.2) and also

$$\mathrm{id}\colon G(\Omega) \hookrightarrow C(\Omega) \hookrightarrow L_1(\Omega) \tag{5.6}$$

must be interpreted as in connection with (4.2). In particular, (5.4), (5.5) make sense. By Definition 4.1 (ii) one has

$$g_k^{\text{lin}}\big(\mathrm{id}\colon G(\Omega) \hookrightarrow L_1(\Omega)\big) = g_k^{\text{lin}}\big(G(\Omega)\big), \quad k \in \mathbb{N}. \tag{5.7}$$

In other words, $g_k^{\text{lin}}\big(G(\Omega)\big)$ are special linear sampling numbers. Assuming that $\{x^l\}_{l=1}^k$ and $\{h_l\}_{l=1}^k$ are chosen optimally in (5.4) then it follows from (5.5) with $a_l = \int_\Omega h_l(x)\,\mathrm{d}x$ that

$$\mathrm{Int}_k\big(G(\Omega)\big) \le g_k^{\text{lin}}\big(G(\Omega)\big), \quad k \in \mathbb{N}. \tag{5.8}$$

However it comes out that in many cases this estimate is an equivalence.

Remark 5.3. Based on Faber bases we identify in Sections 5.2-5.4 $G(\Omega)$ with spaces $A^s_{pq}(I)$ on the unit interval I, with spaces $S^r_{pq}B(\mathbb{Q}^n)$, $S^1_pW(\mathbb{Q}^n)$ having dominating mixed smoothness on cubes $\mathbb{Q}^n$, and with their logarithmic generalisations $S^{r,b}_{pq}B(\mathbb{Q}^n)$, $S^{r,(b)}_{pq}B(\mathbb{Q}^n)$. This gives the possibility to extend these considerations to some spaces on bounded cellular domains in $\mathbb{R}^n$. This will not be done here in detail. But we outline this promising approach in case of planar domains. However first we glance at numerical integration of functions belonging to isotropic spaces in domains in $\mathbb{R}^n$, referring to previous assertions.

5.1.2 Integration in Lipschitz domains

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, and let $A^s_{pq}(\Omega)$ be the isotropic spaces as introduced in Definition 1.24 (i) by restriction of $A^s_{pq}(\mathbb{R}^n)$ to Ω. Here $A \in \{B, F\}$. Then

$$A^s_{pq}(\Omega) \hookrightarrow C(\Omega) \hookrightarrow L_1(\Omega) \quad \text{if } 0 < p, q \le \infty \text{ and } s > n/p, \tag{5.9}$$

($p < \infty$ for F-spaces), (4.15), (4.16), is a special case of (5.6). Let $g^{\mathrm{lin}}_k\big(A^s_{pq}(\Omega)\big)$ and $\mathrm{Int}_k\big(A^s_{pq}(\Omega)\big)$ be as in Definition 5.1. Recall that $a_+ = \max(a, 0)$ for $a \in \mathbb{R}$.

Theorem 5.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$, according to Definition 1.26 (ii) or a bounded interval in $\mathbb{R}$. Let $0 < p, q \le \infty$ ($p < \infty$ for F-spaces) and $s > n/p$. Then*

$$g^{\mathrm{lin}}_k\big(A^s_{pq}(\Omega)\big) \sim \mathrm{Int}_k\big(A^s_{pq}(\Omega)\big) \sim k^{-\frac{s}{n}+(\frac{1}{p}-1)_+}, \quad k \in \mathbb{N}. \tag{5.10}$$

Proof. By (5.8), (4.19) (and (1.142) with Ω in place of $\mathbb{R}^n$) it follows that

$$\mathrm{Int}_k\big(A^s_{pq}(\Omega)\big) \le g^{\mathrm{lin}}_k\big(A^s_{pq}(\Omega)\big) \sim k^{-\frac{s}{n}+(\frac{1}{p}-1)_+}, \quad k \in \mathbb{N}. \tag{5.11}$$

It remains to prove that

$$\mathrm{Int}_k\big(B^s_{pq}(\Omega)\big) \ge c\, k^{-\frac{s}{n}+(\frac{1}{p}-1)_+} \quad \text{for some } c > 0 \text{ and all } k \in \mathbb{N}. \tag{5.12}$$

Let $k = 2^{jn}$, $j \in \mathbb{N}$. Let $\varphi \in D(\mathbb{R}^n)$ be a non-negative function with compact support near the origin and $\varphi(0) > 0$. Let $\{x^l\}^k_{l=1} \subset \Omega$ and $\{a_l\}^k_{l=1} \subset \mathbb{C}$ be given. Then for suitably chosen points $\{y^l\}^k_{l=1} \subset \Omega$ (and a sufficiently small support of φ in dependence on Ω),

$$f^j(x) = \sum_{l=1}^{k} \varphi\big(2^j(x - y^l)\big) \in D(\Omega), \quad f^j(x^l) = 0. \tag{5.13}$$

Furthermore we may assume that there is a number $C > 0$ such that for all $j \in \mathbb{N}$,

$$\int_\Omega f^j(x)\,\mathrm{d}x = C. \tag{5.14}$$

Let $p \ge 1$. We interpret (5.13) as an atomic representation in $B^s_{pq}(\mathbb{R}^n)$. Then it follows from Theorem 1.7 that

$$\|f^j \,|B^s_{pq}(\Omega)\| \le \|f^j \,|B^s_{pq}(\mathbb{R}^n)\| \le c\, 2^{j(s-\frac{n}{p})}\Big(\sum_{l=1}^{2^{jn}} 1\Big)^{1/p} = c\, 2^{js}. \tag{5.15}$$

Inserted in (5.5) one obtains that

$$\mathrm{Int}_k\big(B^s_{pq}(\Omega)\big) \ge c\, k^{-s/n} \quad \text{for some } c > 0 \text{ and all } k \in \mathbb{N}. \tag{5.16}$$

Let $p < 1$. Then we reduce (5.13) to one term, hence $f^j(x) = \varphi\big(2^j(x - y^1)\big)$. One has for some $C > 0$,

$$\int_\Omega f^j(x)\,\mathrm{d}x = C\,2^{-jn}, \quad \|f^j\,|B^s_{pq}(\Omega)\| \le c\,2^{j(s-\frac{n}{p})}, \quad j \in \mathbb{N}. \tag{5.17}$$

One obtains the counterpart of (5.16),

$$\mathrm{Int}_k\big(B^s_{pq}(\Omega)\big) \ge c\,k^{-\frac{s}{n}+\frac{1}{p}-1} \quad \text{for some } c > 0 \text{ and all } k \in \mathbb{N}. \tag{5.18}$$

Now (5.16), (5.18) prove (5.12). □

Remark 5.5. We inserted Theorem 5.4 for sake of completeness complementing corresponding assertions about sampling numbers in isotropic spaces according to Section 4.1.2. Numerical integration for Hölder–Zygmund and Sobolev spaces especially in intervals and cubes has a long history going back to the 1950s. The early history may be found in [NoW08, p. 152]. Detailed references in particular to the Russian literature up to the 1980s are given in [Nov88, p. 37]. Otherwise we are more interested here in explicit (quadrature and cubature) formulas based on Faber expansions. The proof of (5.12) coincides essentially with corresponding arguments in [NoT06, pp. 348/349] in connection with sampling numbers.

5.1.3 A comment on integration in E-thick domains

Numerical integration in $\mathbb{R}^n$ is usually restricted to bounded Lipschitz domains with a preference for cubes. What about bounded domains with, say, fractal boundary? One may think about the snowflake curve in $\mathbb{R}^2$. One can also extend Definition 5.1 to unbounded domains Ω in $\mathbb{R}^n$ with $|\Omega| < \infty$. This ensures (5.6). We mention here a distinguished class of domains beyond bounded Lipschitz domains where one can say something about numerical integration. But we are very brief and restrict ourselves to comments and references.

Let $l(Q)$ be the side-length of a (finite) cube Q in $\mathbb{R}^n$ with sides parallel to the axes of coordinates. Let Ω be a (non-empty) domain in $\mathbb{R}^n$ with $\Omega \neq \mathbb{R}^n$ and $\Gamma = \partial\Omega$. Then Ω is said to be *E-thick* (exterior thick) if one finds for any *interior* cube $Q^i \subset \Omega$ with

$$l(Q^i) \sim 2^{-j}, \quad \mathrm{dist}\,(Q^i, \Gamma) \sim 2^{-j}, \quad j \ge j_0 \in \mathbb{N}, \tag{5.19}$$

a *complementing exterior* cube $Q^e \subset \Omega^c = \mathbb{R}^n \setminus \Omega$ with

$$l(Q^e) \sim 2^{-j}, \quad \mathrm{dist}\,(Q^e, \Gamma) \sim \mathrm{dist}\,(Q^i, Q^e) \sim 2^{-j}, \quad j \ge j_0 \in \mathbb{N}. \tag{5.20}$$

We refer to [T08, Section 3.1] where we discussed in detail classes of domains in $\mathbb{R}^n$ including many examples. In particular, a planar domain with the closed snowflake (Koch curve) as boundary is E-thick. Bounded Lipschitz domains in $\mathbb{R}^n$ are E-thick. In [T08, Section 3.2] we described wavelet expansions for some spaces $A^s_{pq}(\Omega)$ and $\widetilde{A}^s_{pq}(\Omega)$ according to Definition 1.24 in E-thick domains in $\mathbb{R}^n$. This is the basis to complement Theorem 5.4 as follows.

Proposition 5.6. *Let Ω be an E-thick domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, with $|\Omega| < \infty$. Let $0 < p, q \le \infty$ ($p < \infty$ for F-spaces) and $s > n/p$. Then*

$$g_k^{\text{lin}}\big(\widetilde{A}^s_{pq}(\Omega)\big) \sim \text{Int}_k\big(\widetilde{A}^s_{pq}(\Omega)\big) \sim k^{-\frac{s}{n}+(\frac{1}{p}-1)_+}, \quad k \in \mathbb{N}. \tag{5.21}$$

Proof. First we remark that

$$\text{Int}_k\big(\widetilde{A}^s_{pq}(\Omega)\big) \le g_k^{\text{lin}}\big(\widetilde{A}^s_{pq}(\Omega)\big) \sim k^{-\frac{s}{n}+(\frac{1}{p}-1)_+}, \quad k \in \mathbb{N}, \tag{5.22}$$

is the counterpart of (5.11). The first inequality is covered by (5.6), (5.8). The equivalence in (5.22) follows from (5.7) and [T08, Theorem 6.87(ii), p. 241], where we referred for a detailed proof to [Tri07, Theorem 23 (ii), pp. 488/489]. This will not be repeated here. The counterpart of (5.12),

$$\text{Int}_k\big(\widetilde{B}^s_{pq}(\Omega)\big) \ge c\, k^{-\frac{s}{n}+(\frac{1}{p}-1)_+} \quad \text{for some } c > 0 \text{ and all } k \in \mathbb{N}, \tag{5.23}$$

can be obtained in the same way as in the proof of Theorem 5.4. □

5.2 Integration in intervals

5.2.1 Main assertions

Let $I = (0,1)$ be the unit interval on $\mathbb{R}$. We specify (5.9) by

$$A^s_{pq}(I) \hookrightarrow C(I) \hookrightarrow L_1(I) \quad \text{if} \quad 0 < p, q \le \infty, \quad s > 1/p, \tag{5.24}$$

where again $A \in \{B, F\}$ (with $p < \infty$ for F-spaces). Let now $g_k^{\text{lin}}\big(A^s_{pq}(I)\big)$ and $\text{Int}_k\big(A^s_{pq}(I)\big)$ be as in Definition 5.1. So far we have (5.10). However this is not constructive. But for some spaces we are now in a much better comfortable position. We rely on Faber expansions as used in the preceding Chapter 4 in connection with sampling. For sake of convenience we recall some basic ingredients. Let

$$\big\{v_0, v_1, v_{jm} : j \in \mathbb{N}_0;\ m = 0, \ldots, 2^j - 1\big\} \tag{5.25}$$

with

$$v_0(x) = 1 - x, \quad v_1(x) = x, \quad \text{where } 0 \le x \le 1, \tag{5.26}$$

and

$$v_{jm}(x) = \begin{cases} 2^{j+1}(x - 2^{-j}m) & \text{if } 2^{-j}m \le x < 2^{-j}m + 2^{-j-1}, \\ 2^{j+1}(2^{-j}(m+1) - x) & \text{if } 2^{-j}m + 2^{-j-1} \le x < 2^{-j}(m+1), \\ 0 & \text{otherwise,} \end{cases} \tag{5.27}$$

$0 \le x \le 1$, be the Faber system in I according to (3.1)–(3.3), Figure 2.1, p. 64. Let

$$\lambda_{jm}(f) = -\tfrac{1}{2}\big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m), \quad j \in \mathbb{N}_0, \quad m = 0, \ldots, 2^j - 1. \tag{5.28}$$

By Theorem 3.1 any $f \in A^s_{pq}(I)$ with (5.24) can be expanded by

$$f(x) = f(0)\, v_0(x) + f(1)\, v_1(x) + \sum_{j=0}^{\infty} \sum_{m=0}^{2^j-1} \lambda_{jm}(f)\, v_{jm}(x), \quad x \in I. \tag{5.29}$$

If, in addition, p, q, s are restricted as in Theorem 3.1 then J in (3.22) is the indicated isomorphic map onto related sequence spaces. Let

$$S^J f(x) = f(0)\, v_0(x) + f(1)\, v_1(x) + \sum_{j=0}^{J} \sum_{m=0}^{2^j-1} \lambda_{jm}(f)\, v_{jm}(x), \quad x \in I, \tag{5.30}$$

be the same beginning of (5.29) as in (4.63). Then

$$\begin{aligned} \int_I S^J f(x)\,\mathrm{d}x &= \tfrac{1}{2} f(0) + \tfrac{1}{2} f(1) - \sum_{j=0}^{J} 2^{-j-2} \sum_{m=0}^{2^j-1} \big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m) \\ &= \sum_{l=0}^{2^{J+1}} a^J_l\, f\big(2^{-J-1} l\big). \end{aligned} \tag{5.31}$$

In (5.39) we calculate the numbers a^J_l explicitly. Recall that $b_+ = \max(b, 0)$, $b \in \mathbb{R}$.

Theorem 5.7. *Let $0 < p, q \le \infty$ ($p < \infty$ for F-spaces) and*

$$\frac{1}{p} < s < 1 + \frac{1}{p}. \tag{5.32}$$

Then

$$g^{\mathrm{lin}}_k\big(A^s_{pq}(I)\big) \sim \mathrm{Int}_k\big(A^s_{pq}(I)\big) \sim k^{-s+(\frac{1}{p}-1)_+}, \quad k \in \mathbb{N}. \tag{5.33}$$

Furthermore, there is a number $c > 0$ such that for all $J \in \mathbb{N}$,

$$\Big| \int_I f(x)\,\mathrm{d}x - \sum_{l=0}^{2^{J+1}} a^J_l\, f\big(2^{-J-1} l\big) \Big| \le c\, \mathrm{Int}_{1+2^{J+1}}\big(A^s_{pq}(I)\big)\, \|f \,|A^s_{pq}(I)\|, \tag{5.34}$$

with a^J_l as in (5.31).

Proof. Step 1. Let $p \ge 1$ and $k = 1 + 2^{J+1}$. Then it follows from (5.7), (5.8), Theorem 4.11 (and (4.54)) that

$$\mathrm{Int}_k\big(A^s_{pq}(I)\big) \le g^{\mathrm{lin}}_k\big(A^s_{pq}(I)\big) \sim k^{-s}. \tag{5.35}$$

The left-hand side can be estimated from below by ck^{-s}, $c > 0$, as in the proof of Theorem 5.4. This proves (5.33). The assertion (5.34) follows from (4.68) and (5.31).

Step 2. Let $p < 1$ and $\sigma - 1 = s - \frac{1}{p}$. Then

$$A^s_{pq}(I) \hookrightarrow A^\sigma_{1,q}(I). \tag{5.36}$$

We apply Step 1 to $A^\sigma_{1,q}(I)$ and obtain

$$\mathrm{Int}_k\big(A^s_{pq}(I)\big) \le g^{\mathrm{lin}}_k\big(A^s_{pq}(I)\big) \le c\, g^{\mathrm{lin}}_k\big(A^\sigma_{1,q}(I)\big) \sim k^{-s+\frac{1}{p}-1}. \tag{5.37}$$

The left-hand side of (5.37) can be estimated from below by $c\,k^{-s+\frac{1}{p}-1}$ with some $c > 0$ as in the proof of Theorem 5.4. Finally (5.34) follows from (4.68). □

Corollary 5.8. *Let p, q, s be as in Theorem 5.7. Then there is a number $c > 0$ such that for all $J \in \mathbb{N}$ and all $f \in A^s_{pq}(I)$,*

$$\begin{aligned} &\Big| \int_I f(x)\,\mathrm{d}x - 2^{-J-2}\big(f(0) + f(1)\big) - 2^{-J-1} \sum_{l=1}^{2^{J+1}-1} f\big(l\,2^{-J-1}\big)\Big| \\ &\quad \le c\,\mathrm{Int}_{1+2^{J+1}}\big(A^s_{pq}(I)\big)\,\|f\,|A^s_{pq}(I)\| \\ &\quad \sim 2^{-Js+J(\frac{1}{p}-1)_+}\,\|f\,|A^s_{pq}(I)\|. \end{aligned} \tag{5.38}$$

Proof. By Theorem 5.7 and (5.31) it remains to show that for $J \in \mathbb{N}_0$,

$$\begin{aligned} &\sum_{l=0}^{2^{J+1}} a^J_l\, f\big(2^{-J-1}l\big) \\ &\quad = \tfrac{1}{2} f(0) + \tfrac{1}{2} f(1) \\ &\qquad - \sum_{j=0}^{J} 2^{-j-2} \sum_{m=0}^{2^j-1} \Big[f\big(2^{-j}m\big) + f\big(2^{-j}(m+1)\big) - 2f\big(2^{-j}m + 2^{-j-1}\big)\Big] \\ &\quad = 2^{-J-1}\Big[\tfrac{1}{2} f(0) + \tfrac{1}{2} f(1) + \sum_{l=1}^{2^{J+1}-1} f\big(l\,2^{-J-1}\big)\Big]. \end{aligned} \tag{5.39}$$

We prove this equality by induction. The case $J = 0$ can be checked directly. Then one obtains by iteration for $J \in \mathbb{N}$ that

$$\begin{aligned} \sum_{l=0}^{2^{J+1}} a^J_l\, f\big(2^{-J-1}l\big) &= \sum_{l=0}^{2^J} a^{J-1}_l\, f\big(2^{-J}l\big) \\ &\quad - 2^{-J-2} \sum_{m=0}^{2^J-1} \big[f\big(2^{-J}m\big) + f\big(2^{-J}(m+1)\big) - 2f\big(2^{-J}m + 2^{-J-1}\big)\big] \end{aligned} \tag{5.40}$$

$$= 2^{-J-2}\big(f(0) + f(1)\big) + \sum_{m=1}^{2^J-1} 2^{-J-1} f\big(2^{-J}m\big) + \sum_{m=0}^{2^J-1} 2^{-J-1}\, f\big(2^{-J}m + 2^{-J-1}\big)$$

$$= 2^{-J-1}\Big[\tfrac{1}{2} f(0) + \tfrac{1}{2} f(1) + \sum_{l=1}^{2^{J+1}-1} f\big(2^{-J-1}l\big)\Big].$$

This proves (5.39) and, hence, the corollary. □

5.2.2 Comments and inequalities

We discuss the assertions from Theorem 5.7 and Corollary 5.8 and have a closer look at a few special cases. Let again $I = (0,1)$ be the unit interval on $\mathbb{R}$ and let

$$\big(\Delta^M_{h,I} f\big)(x) = \begin{cases} \big(\Delta^M_h f\big)(x) & \text{if } x + lh \in I \text{ for } l = 0, \ldots, M, \\ 0 & \text{otherwise,} \end{cases} \tag{5.41}$$

where $x \in I$, $M \in \mathbb{N}$ and $h \in \mathbb{R}$, be the same adapted differences as in (1.287) based on (1.21). Let

$$0 < p, q \le \infty, \qquad \max\Big(\frac{1}{p}, 1\Big) - 1 < s < M \in \mathbb{N}. \tag{5.42}$$

Then

$$\|f \,|B^s_{pq}(I)\|_M = \|f\,|L_p(I)\| + \Big(\int_0^1 h^{-sq}\, \|\Delta^M_{h,I} f\,|L_p(I)\|^q\, \frac{\mathrm{d}h}{h}\Big)^{1/q} \tag{5.43}$$

is an equivalent quasi-norm in $B^s_{pq}(I)$. This coincides with (1.290) where one finds also some relevant references.

Corollary 5.9. *Let* $0 < p, q \le \infty$ *(*$p < \infty$ *for* F*-spaces) and*

$$\frac{1}{p} < s < 1 + \frac{1}{p}, \qquad s < M \in \mathbb{N}. \tag{5.44}$$

Then

$$\begin{aligned} &\Big|\sum_{m=0}^{2^J-1} \int_{m2^{-J}}^{(m+1)2^{-J}} \big(f(x) - \tfrac{1}{2} f\big(2^{-J}m\big) - \tfrac{1}{2} f\big(2^{-J}(m+1)\big)\big)\,\mathrm{d}x\Big| \\ &= \Big|\int_I f(x)\,\mathrm{d}x - 2^{-J-1}\big(f(0) + f(1)\big) - 2^{-J} \sum_{l=1}^{2^J-1} f\big(l\,2^{-J}\big)\Big| \\ &\le c\, 2^{-Js + J(\frac{1}{p}-1)_+} \inf_{d \in \mathbb{C}} \|f + d\,|A^s_{pq}(I)\| \end{aligned} \tag{5.45}$$

for some $c > 0$, *all* $J \in \mathbb{N}$ *and all* $f \in A^s_{pq}(I)$. *If* $A = B$ *then*

$$\Big| \int_I f(x)\,\mathrm{d}x - 2^{-J-1}\big(f(0)+f(1)\big) - 2^{-J} \sum_{l=1}^{2^J-1} f\big(l\,2^{-J}\big)\Big| \le c\,2^{-Js+J(\frac{1}{p}-1)_+} \Big[\inf_{d\in\mathbb{C}} \|f+d\,|L_p(I)\| + \Big(\int_0^1 h^{-sq} \|\Delta^M_{h,I} f\,|L_p(I)\|^q \frac{\mathrm{d}h}{h}\Big)^{1/q}\Big] \tag{5.46}$$

(*with the usual modification if* $q = \infty$).

Proof. The first equality in (5.45) is a matter of direct arguments. It remains unchanged if one replaces f by $f + d$ with $d \in \mathbb{C}$. Then the estimate in (5.45) follows from (5.38) (with $J - 1$ in place of J). The estimate (5.46) is now a consequence of (5.43). □

One may ask for further simplifications and examples. Let $s < 1$ in (5.42), (5.43). Then one can choose $M = 1$. With $\Delta^1_{h,I} = \Delta_{h,I} f$ one has

$$\|f\,|B^s_{pq}(I)\| \sim \Big|\int_I f(x)\,\mathrm{d}x\Big| + \Big(\int_0^1 h^{-sq} \|\Delta_{h,I} f\,|L_p(I)\|^q \frac{\mathrm{d}h}{h}\Big)^{1/q} \tag{5.47}$$

(equivalent quasi-norms). Similarly

$$\|f\,|W^1_p(I)\| = \Big|\int_I f(x)\,\mathrm{d}x\Big| + \|f'\,|L_p(I)\|, \quad 1 < p < \infty, \tag{5.48}$$

is an equivalent norm in $W^1_p(I)$. This is well known. But it can also be proved by the same arguments as in (3.44), (3.45).

Corollary 5.10. (i) *Let* $1 < p \le \infty$, $0 < q \le \infty$, *and* $\frac{1}{p} < s < 1$. *Then*

$$\Big| \int_I f(x)\,\mathrm{d}x - 2^{-J} \sum_{m=0}^{2^J-1} f\big(2^{-J}m\big)\Big| \le c\,2^{-Js} \Big(\int_0^1 h^{-sq} \|\Delta_{h,I} f\,|L_p(I)\|^q \frac{\mathrm{d}h}{h}\Big)^{1/q} \tag{5.49}$$

(*with the usual modification if* $q = \infty$) *for some* $c > 0$, *all* $J \in \mathbb{N}$ *and all* $f \in B^s_{pq}(I)$.

(ii) *Let* $1 < p < \infty$. *Then*

$$\Big| \int_I f(x)\,\mathrm{d}x - 2^{-J} \sum_{m=0}^{2^J-1} f\big(2^{-J}m\big)\Big| \le c\,2^{-J} \|f'\,|L_p(I)\| \tag{5.50}$$

for some $c > 0$, *all* $J \in \mathbb{N}$ *and all* $f \in W^1_p(I)$.

Proof. If $s \le 1$ then it follows from (5.24) that one can replace $\frac{1}{2}f(0) + \frac{1}{2}f(1)$ in (5.45) by $f(0)$ (or $f(1)$) and again f by $f + d$ with $d \in \mathbb{C}$. Then one obtains (5.49) from (5.47) and (5.50) from (5.48). □

Our approach relies on the Faber expansion (5.29). If p, q, s as in Theorem 5.7 and $f \in A^s_{pq}(I)$ then it follows by integration the absolutely convergent representation

$$\int_I f(x)\,\mathrm{d}x = \tfrac{1}{2} f(0) + \tfrac{1}{2} f(1) - \sum_{j=0}^{\infty} 2^{-j-2} \sum_{m=0}^{2^j-1} \big(\Delta^2_{2^{-j-1}} f\big)(2^{-j}m). \tag{5.51}$$

The assertions about Faber bases give the possibility to control the rate of convergence. This has been done in Theorem 5.7 and in the Corollaries 5.8-5.10. We do not know to which extent (5.51) and the above consequences are known. We have no references. But there are a few cases in which the above inequalities can be proved by rather elementary means. Here are two examples.

Example 1. Let $f \in C^s(I) = \mathcal{C}^s(I) = B^s_{\infty\infty}(I)$ with $0 < s < 1$ (Hölder spaces). Then

$$\begin{aligned} \Big| \int_I f(x)\,\mathrm{d}x - 2^{-J} \sum_{m=0}^{2^J-1} f\big(2^{-J}m\big) \Big| &\le \sum_{m=0}^{2^J-1} \int_{m2^{-J}}^{(m+1)2^{-J}} |f(x) - f(2^{-J}m)|\,\mathrm{d}x \\ &\le 2^{-sJ} \sup_{x,y \in I} \frac{|f(x) - f(y)|}{|x-y|^s}. \end{aligned} \tag{5.52}$$

This is a special case of (5.49). It covers also $f \in \mathrm{Lip}(I)$ with $s = 1$.

Example 2. Let $f \in W^1_p(I)$ with $1 \le p < \infty$ (including $p = 1$). Then one has for $2^{-J}m < x < 2^{-J}(m+1)$ with $m = 0, \ldots, 2^J - 1$,

$$\begin{aligned} |f(x) - f(2^{-J}m)| &= \Big| \int_{2^{-J}m}^{x} f'(y)\,\mathrm{d}y \Big| \\ &\le \Big(\int_{2^{-J}m}^{2^{-J}(m+1)} |f'(y)|^p\,\mathrm{d}y \Big)^{1/p} (x - 2^{-J}m)^{1/p'}, \end{aligned} \tag{5.53}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. It follows that

$$\begin{aligned} &\sum_{m=0}^{2^J-1} \int_{2^{-J}m}^{2^{-J}(m+1)} |f(x) - f(2^{-J}m)|\,\mathrm{d}x \\ &\qquad \le \frac{p'}{p'+1} \sum_{m=0}^{2^J-1} 2^{-J(\frac{1}{p'}+1)} \Big(\int_{2^{-J}m}^{2^{-J}(m+1)} |f'(y)|^p\,\mathrm{d}y \Big)^{1/p} \\ &\qquad \le \frac{p'}{p'+1} 2^{-J(\frac{1}{p'}+1)} 2^{J/p'} \Big(\int_0^1 |f'(y)|^p\,\mathrm{d}y \Big)^{1/p} \end{aligned} \tag{5.54}$$

and

$$\Big| \int_I f(x)\,\mathrm{d}x - 2^{-J} \sum_{m=0}^{2^J-1} f(2^{-J}m) \Big| \le \frac{p'}{p'+1} 2^{-J} \|f' \,|L_p(I)\|. \tag{5.55}$$

This coincides with (5.50) where $c = \frac{p'}{p'+1}$.

5.3 Multivariate integration

5.3.1 Integration in squares

In (5.5) we introduced the integral numbers $\mathrm{Int}_k\big(G(\Omega)\big)$ for spaces $G(\Omega)$ with (5.2) in bounded domains Ω in $\mathbb{R}^n$. We have Theorem 5.4 for isotropic spaces $A^s_{pq}(\Omega)$ on bounded Lipschitz domains. But there are no constructive algorithms for optimal approximations according to (5.5). The situation is much better if $\Omega = I = (0,1)$ is the unit interval on $\mathbb{R}$. Then we have Theorem 5.7 and Corollary 5.8 based on the Faber expansions (5.25)–(5.29). Now we are doing the same in higher dimensions always guided by the attempt to preserve the error bound in (5.33) for spaces on I of smoothness s in higher dimensions (maybe at the expense of log-terms). First we deal with integral numbers and constructive algorithms for spaces with dominating mixed smoothness in the unit square

$$\mathbb{Q}^2 = \big\{x = (x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < 1\big\}. \tag{5.56}$$

For sake of convenience we repeat a few notation and basic assertions. Let

$$\{v_0, v_1, v_{jm} : j \in \mathbb{N}_0;\ m = 0, \ldots, 2^j - 1\} \tag{5.57}$$

be the Faber system of the unit interval $I = (0,1)$ according to (5.25)–(5.27). Let

$$\big\{v_{km} : k \in \mathbb{N}^2_{-1},\ m \in \mathbb{P}^F_k\big\} \tag{5.58}$$

be the same Faber system in $\mathbb{Q}^2$ as in (3.62)=(4.116) consisting of the functions

$$v_{km}(x) = \begin{cases} v_{m_1}(x_1)\, v_{m_2}(x_2) \\ \qquad \text{if } k = (-1,-1);\ m_1 \in \{0,1\},\ m_2 \in \{0,1\}, \\ v_{m_1}(x_1)\, v_{k_2 m_2}(x_2) \\ \qquad \text{if } k = (-1,k_2),\ k_2 \in \mathbb{N}_0;\ m_1 \in \{0,1\},\ m_2 = 0, \ldots, 2^{k_2} - 1, \\ v_{k_1 m_1}(x_1)\, v_{m_2}(x_2) \\ \qquad \text{if } k = (k_1,-1),\ k_1 \in \mathbb{N}_0;\ m_1 = 0, \ldots, 2^{k_1} - 1,\ m_2 \in \{0,1\}, \\ v_{k_1 m_1}(x_1)\, v_{k_2 m_2}(x_2) \\ \qquad \text{if } k \in \mathbb{N}^2_0;\ m_l = 0, \ldots, 2^{k_l} - 1;\ l = 1, 2, \end{cases} \tag{5.59}$$

$x = (x_1, x_2) \in \mathbb{Q}^2$, where

$$\mathbb{P}^F_k = \big\{m \in \mathbb{Z}^2 \text{ with } m \text{ as in (5.59)}\big\}, \quad k \in \mathbb{N}^2_{-1}. \tag{5.60}$$

Furthermore we recall the mixed differences (3.74)–(3.77) = (4.119)–(4.122),

$$d^2_{km}(f) = f(m_1, m_2) \quad \text{if } k = (-1,-1),\, m_1 \in \{0,1\},\, m_2 \in \{0,1\}, \tag{5.61}$$

$$\begin{aligned} d^2_{km}(f) &= -\tfrac{1}{2}\Delta^2_{2^{-k_2-1},2}\, f\big(m_1, 2^{-k_2}m_2\big) \\ &\qquad \text{if } k = (-1,k_2),\, k_2 \in \mathbb{N}_0,\, m_1 \in \{0,1\},\, m_2 = 0,\dots,2^{k_2}-1, \end{aligned} \tag{5.62}$$

$$\begin{aligned} d^2_{km}(f) &= -\tfrac{1}{2}\Delta^2_{2^{-k_1-1},1}\, f\big(2^{-k_1}m_1, m_2\big) \\ &\qquad \text{if } k = (k_1,-1),\, k_1 \in \mathbb{N}_0,\, m_2 \in \{0,1\},\, m_1 = 0,\dots,2^{k_1}-1, \end{aligned} \tag{5.63}$$

$$\begin{aligned} d^2_{km}(f) &= \tfrac{1}{4}\Delta^{2,2}_{2^{-k_1-1},2^{-k_2-1}}\, f\big(2^{-k_1}m_1, 2^{-k_2}m_2\big) \\ &\qquad \text{if } k \in \mathbb{N}^2_0,\, m_l = 0,\dots,2^{k_l}-1;\, l = 1,2, \end{aligned} \tag{5.64}$$

where the (mixed) differences have the same meaning as in (3.72), (3.73). Let $S^r_{pq}B(\mathbb{Q}^2)$ with p,q,r as in (4.123) and $S^1_pW(\mathbb{Q}^2)$ with $1 < p < \infty$ be the same spaces with dominating mixed smoothness as in Theorems 3.13, 3.16 and Corollary 3.14. We added in (4.127) a few limiting cases resulting in

$$S^r_{pq}\mathfrak{B}(\mathbb{Q}^2), \quad 0 < p,q \le \infty,\ \tfrac{1}{p} < r < 1 + \min\big(\tfrac{1}{p}, 1\big), \tag{5.65}$$

with (4.128). If $f \in S^r_{pq}\mathfrak{B}(\mathbb{Q}^2)$ or $f \in S^1_pW(\mathbb{Q}^2)$ then f can be represented by the series

$$f = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(f)\, v_{km} \tag{5.66}$$

which converges absolutely and hence unconditionally in $C(\mathbb{Q}^2)$ and (as a consequence) in $L_1(\mathbb{Q}^2)$. Furthermore one has (4.127) and (3.92). Integration over $\mathbb{Q}^2$ results in an absolutely convergent series. One obtains by (5.59) and (5.61)–(5.64) the explicit remarkable representation

$$\begin{aligned} \int_{\mathbb{Q}^2} f(x)\,\mathrm{d}x = &\sum_{m_1,m_2 \in \{0,1\}} 2^{-2} f(m_1,m_2) \\ &- \sum_{k_2=0}^{\infty} \sum_{m_2=0}^{2^{k_2}-1} \sum_{m_1=0}^{1} 2^{-k_2-3}\Delta^2_{2^{-k_2-1},2}\, f\big(m_1, 2^{-k_2}m_2\big) \\ &- \sum_{k_1=0}^{\infty} \sum_{m_1=0}^{2^{k_1}-1} \sum_{m_2=0}^{1} 2^{-k_1-3}\Delta^2_{2^{-k_1-1},1}\, f\big(2^{-k_1}m_1, m_2\big) \\ &+ \sum_{k \in \mathbb{N}^2_0} \sum_{m_1=0}^{2^{k_1}-1} \sum_{m_2=0}^{2^{k_2}-1} 2^{-k_1-k_2-4}\Delta^{2,2}_{2^{-k_1-1},2^{-k_2-1}}\, f\big(2^{-k_1}m_1, 2^{-k_2}m_2\big). \end{aligned} \tag{5.67}$$

We split f as in (4.134), (4.135), hence

$$\begin{aligned} f &= \sum_{k_1+k_2 \le K} \sum_{m \in \mathbb{P}_k^F} d_{km}^2(f)\, v_{km} + \sum_{k_1+k_2 > K} \sum_{m \in \mathbb{P}_k^F} d_{km}^2(f)\, v_{km} \\ &= M^K f + R^K f, \quad K \in \mathbb{N}. \end{aligned} \tag{5.68}$$

Integration over $\mathbb{Q}^2$ gives a corresponding decomposition of (5.67),

$$\int_{\mathbb{Q}^2} f(x)\,dx = \int_{\mathbb{Q}^2} (M^K f)(x)\,dx + \int_{\mathbb{Q}^2} (R^K f)(x)\,dx = \overline{M}^K f + \overline{R}^K f. \tag{5.69}$$

Recall that $a_+ = \max(a,0)$ if $a \in \mathbb{R}$.

Proposition 5.11. (i) *Let*

$$0 < p,q \le \infty, \quad \frac{1}{p} < r < 1 + \min\Big(\frac{1}{p}, 1\Big). \tag{5.70}$$

Then there is a constant $c > 0$ such that for all $K \in \mathbb{N}$ and all $f \in S^r_{pq}\mathfrak{B}(\mathbb{Q}^2)$,

$$\Big| \int_{\mathbb{Q}^2} f(x)\,dx - \overline{M}^K f \Big| \le c\, 2^{-Kr + K(\frac{1}{p}-1)_+}\, K^{(1-\frac{1}{q})_+}\, \| f \,|S^r_{pq}\mathfrak{B}(\mathbb{Q}^2)\|. \tag{5.71}$$

(ii) *Let $1 < p < \infty$. Then there is a constant $c > 0$ such that for all $K \in \mathbb{N}$ and all $f \in S^1_p W(\mathbb{Q}^2)$,*

$$\Big| \int_{\mathbb{Q}^2} f(x)\,dx - \overline{M}^K f \Big| \le c\, 2^{-K}\, K^{1/2}\, \| f \,|S^1_p W(\mathbb{Q}^2)\|. \tag{5.72}$$

Proof. We prove part (i). Obviously,

$$\Big| \int_{\mathbb{Q}^2} f(x)\,dx - \overline{M}^K f \Big| \le \| R^K f \,|L_1(\mathbb{Q}^2)\|. \tag{5.73}$$

Then one obtains (5.71) from (4.142), (4.144) with $u = 1$. Now part (ii) with $1 < p \le 2$ follows from the right-hand side of (4.159). If $2 < p < \infty$ then one applies afterwards (4.158). □

Remark 5.12. This is the counterpart of Corollary 5.10 and the examples at the end of Section 5.2.2. It might be useful to look at a few cases.

1. Sobolev spaces. Let $1 < p < \infty$ and $f \in S^1_p W(\mathbb{Q}^2)$ with $f|\partial\mathbb{Q}^2 = 0$. Then it follows from (5.67), (5.68), (5.72) and (3.118) that

$$\begin{aligned} &\Big| \int_{\mathbb{Q}^2} f(x)\,dx - \sum_{k_1+k_2 \le K} \sum_{m_1=0}^{2^{k_1}-1} \sum_{m_2=0}^{2^{k_2}-1} 2^{-k_1-k_2-4} \Delta^{2,2}_{2^{-k_1-1},2^{-k_2-1}} f\big(2^{-k_1}m_1, 2^{-k_2}m_2\big) \Big| \\ &\quad \le c\, 2^{-K}\, K^{1/2} \Big\| \frac{\partial^2 f}{\partial x_1 \partial x_2} \,|L_p(\mathbb{Q}^2) \Big\|. \end{aligned} \tag{5.74}$$

2. Besov spaces. Let

$$0 < p \le \infty, \quad \frac{1}{p} < r < 1 + \min\Big(\frac{1}{p}, 1\Big) \tag{5.75}$$

and $f \in S^r_{pp}B(\mathbb{Q}^2)$ $(= S^r_{pp}\mathfrak{B}(\mathbb{Q}^2))$ with $f|\partial\mathbb{Q}^2 = 0$. Then it follows from (5.71) and (3.117) that

$$\begin{aligned}&\Big|\int_{\mathbb{Q}^2} f(x)\,\mathrm{d}x - \sum_{k_1+k_2\le K}\sum_{m_1=0}^{2^{k_1}-1}\sum_{m_2=0}^{2^{k_2}-1} 2^{-k_1-k_2-4}\,\Delta^{2,2}_{2^{-k_1-1},2^{-k_2-1}} f\big(2^{-k_1}m_1, 2^{-k_2}m_2\big)\Big| \\ &\quad\le c\, 2^{-Kr+K(\frac{1}{p}-1)_+}\, K^{(1-\frac{1}{p})_+}\Big(\int_0^1\int_0^1 (h_1h_2)^{-rp}\,\big\|\Delta^{2,2}_{h,\mathbb{Q}^2} f\,|L_p(\mathbb{Q}^2)\big\|^p \frac{\mathrm{d}h}{h_1h_2}\Big)^{1/p}\end{aligned} \tag{5.76}$$

with the usual modification if $p = \infty$ where we have to justify that in case of $f|\partial\mathbb{Q}^2 = 0$ the last term in (3.117) is an equivalent quasi-norm in $S^r_{pp}B(\mathbb{Q}^2)$. For this purpose we first remark that one can replace $\|f\,|L_p(\mathbb{Q}^2)\|$ in (3.117) by $\|\mathrm{tr}_{\partial\mathbb{Q}^2} f\,|L_p(\partial\mathbb{Q}^2)\|$. This is essentially a one-dimensional assertion covered by (3.42), (3.44). But the same argument can be applied afterwards also to $\Delta^2_{h,1,\mathbb{Q}^2} f$ and $\Delta^2_{h,2,\mathbb{Q}^2} f$. After these replacements the right-hand side of (3.117) reduces in case of $f|\partial\mathbb{Q}^2 = 0$ to the last term in (3.117).

Let $g^{\mathrm{lin}}_k\big(G(\Omega)\big)$ and $\mathrm{Int}_k\big(G(\Omega)\big)$ be the numbers introduced in Definition 5.1 with $G(\Omega)$ as in the above Proposition 5.11. Let $l \in \mathbb{N}$, $l \ge 2$ and $K_l \in \mathbb{N}$ with

$$K_l = \log l - \log\log l + \varepsilon, \quad \text{where } 0 \le \varepsilon < 1. \tag{5.77}$$

Recall that $M^{K_l} f$ in (4.177) evaluates $f \in C(\mathbb{Q}^2)$ in $\sim l$ points in $\overline{\mathbb{Q}^2}$. This follows, as mentioned there, from (4.145) = (5.77). The same assertion is valid for

$$\overline{M}^{K_l} f = \int_{\mathbb{Q}^2} \big(M^{K_l} f\big)(x)\,\mathrm{d}x, \quad 2 \le l \in \mathbb{N}. \tag{5.78}$$

We refer also to the explicit formulas (5.67)–(5.69).

Theorem 5.13. (i) *Let*

$$0 < p, q \le \infty, \quad \frac{1}{p} < r < 1 + \Big(\frac{1}{p}, 1\Big). \tag{5.79}$$

Then

$$\begin{aligned} c_1\, l^{-r+(\frac{1}{p}-1)_+}(\log l)^{(1-\frac{1}{q})_+} &\le \mathrm{Int}_l\big(S^r_{pq}\mathfrak{B}(\mathbb{Q}^2)\big) \le g^{\mathrm{lin}}_l\big(S^r_{pq}\mathfrak{B}(\mathbb{Q}^2)\big) \\ &\le c_2\Big(\frac{l}{\log l}\Big)^{-r+(\frac{1}{p}-1)_+}(\log l)^{(1-\frac{1}{q})_+}\end{aligned} \tag{5.80}$$

and

$$\left| \int_{\mathbb{Q}^2} f(x)\,\mathrm{d}x - \overline{M}^{K_l} f \right| \le c \left(\frac{l}{\log l} \right)^{-r+(\frac{1}{p}-1)_+} (\log l)^{(1-\frac{1}{q})_+} \| f \,|S^r_{pq}\mathfrak{B}(\mathbb{Q}^2)\| \tag{5.81}$$

for some $c_1 > 0$, $c_2 > 0$, $c > 0$, *and all* $l \in \mathbb{N}$, $l \ge 2$.

(ii) *Let* $1 < p < \infty$. *Then*

$$c_1 l^{-1} (\log l)^{1/2} \le \mathrm{Int}_l \big(S^1_p W(\mathbb{Q}^2) \big) \le g^{\mathrm{lin}}_l \big(S^1_p W(\mathbb{Q}^2) \big) \le c_2 l^{-1} (\log l)^{3/2} \tag{5.82}$$

and

$$\left| \int_{\mathbb{Q}^2} f(x)\,\mathrm{d}x - \overline{M}^{K_l} f \right| \le c\, l^{-1} (\log l)^{3/2} \, \| f \,|S^1_p W(\mathbb{Q}^2)\| \tag{5.83}$$

for some $c_1 > 0$, $c_2 > 0$, $c > 0$ *and all* $l \in \mathbb{N}$, $l \ge 2$.

Proof. The right-hand sides of (5.80) and (5.82) follow from Theorem 4.15 and Corollary 4.16 (iii) with $u = 1$. The estimates from below in Steps 3 and 4 of the proof of Theorem 4.15 rely on non-negative functions of type (4.149), (4.153). In particular if $u = 1$ then $\| f \,|L_1(\mathbb{Q}^2)\| = \int_{\mathbb{Q}^2} f(x)\,\mathrm{d}x$ applies also to the integrals over these functions. Then the left-hand sides of (5.80) and (5.82) follow from the arguments resulting in the corresponding left-hand sides in Theorem 4.15 and Corollary 4.16. Using in addition (5.8) one obtains (5.80), (5.82). Finally (5.81) and (5.83) follow from Proposition 5.11 and the arguments in the proof of Theorem 4.15. □

Remark 5.14. Similarly as in Theorem 4.15 there remains a gap between the left-hand sides and the right-hand sides of (5.80), (5.82). On the other hand these gaps are not larger for the integral numbers $\mathrm{Int}_k\big(G(\mathbb{Q}^2)\big)$ than for the specific linear sampling numbers $g^{\mathrm{lin}}_k\big(G(\mathbb{Q}^2)\big)$. As for notation we refer to Definition 5.1. One may compare the above Theorem 5.13 with Theorem 5.7 where we dealt with integration in the unit interval $I = (0,1)$ for spaces $A^r_{pq}(I)$. There are several remarkable differences. First, the final assertion (5.34) must now be replaced by (5.81), (5.83) which might be the best possible estimates in the context of our approach. Secondly, one can take (5.33) with $s = r$ as a guide asking for spaces with (main) smoothness r having the same behaviour. This is the point where dominating mixed smoothness enters the stage. By (5.80), (5.82) one has the same behaviour as in (5.33) for the (main) smoothness r $(= s)$ now perturbed by log-terms. This is in sharp contrast to corresponding assertions for isotropic spaces as described in Theorem 5.4. Thirdly, one may ask for the precise behaviour of the integral numbers in (5.80), (5.82). This is related to discrepancy which will be treated later on in higher dimensions. We discussed this point at the end of Section 4.3.3 and took these observations in Section 4.4.1 as a motivation to introduce logarithmic spaces with dominating mixed smoothness. This will be also one of the main points in what follows. But first we extend a few assertions in Theorem 5.13 from two to higher dimensions.

5.3.2 Integration in cubes

Sampling and integration in the square $\mathbb{Q}^2$ is based on the Faber system (5.58)–(5.60), the mixed differences (5.61)–(5.64) and the representation (5.66). All this can be extended from two to higher dimensions. We refer in particular to Section 3.2.5. Let

$$\mathbb{Q}^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_l < 1\} \tag{5.84}$$

be the unit cube in $\mathbb{R}^n$, $n \geq 2$. As in (3.178) we denote by

$$\{v_{km} : k \in \mathbb{N}^n_{-1},\ m \in \mathbb{P}^{F,n}_k\} \tag{5.85}$$

the n-dimensional generalisation of the Faber system (5.58)–(5.60). The n-dimensional versions of the mixed differences in (5.61)–(5.64) are denoted by $d^2_{km}(f)^n$. There are direct counterparts of Theorems 3.13, 3.16 with $\mathbb{Q}^n$ in place of $\mathbb{Q}^2$ and p, q, r as there, based on the expansions

$$f = \sum_{k \in \mathbb{N}^n_{-1}} \sum_{m \in \mathbb{P}^{F,n}_k} d^2_{km}(f)^n\, v_{km}, \tag{5.86}$$

or (3.179), (3.180). As for properties of the Sobolev spaces $S^1_p W(\mathbb{Q}^n)$ we refer to Section 3.2.5. In addition we are interested in the spaces $S^r_{pq}B(\mathbb{Q}^n)$ with p, q, r as in (4.123) and covered by the n-dimensional versions of Theorems 3.13, 3.16. Similarly as in (5.65) we add a few limiting cases,

$$S^r_{pq}\mathfrak{B}(\mathbb{Q}^n), \quad 0 < p, q \leq \infty,\ \tfrac{1}{p} < r < 1 + \min\left(\tfrac{1}{p}, 1\right), \tag{5.87}$$

consisting of all f in (5.86) with

$$\|f\,|S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)\| = \Big(\sum_{k \in \mathbb{N}^n_{-1}} 2^{(k_1+\cdots+k_n)(r-\frac{1}{p})q} \Big(\sum_{m \in \mathbb{P}^{F,n}_k} |d^2_{km}(f)^n|^p \Big)^{q/p} \Big)^{1/q} < \infty \tag{5.88}$$

with the usual modifications if $p = \infty$ and/or $q = \infty$. One may also consult (4.126)–(4.128) and the comments given there. Theorem 4.15 and Corollary 4.16 dealing with sampling numbers can now be extended from two to higher dimensions. This applies also to Proposition 5.11. This will not be done here. But we give an explicit formulation of the n-dimensional generalisations of (5.80), (5.82) and outline the necessary modifications in the related proofs. We specify now $G(\Omega)$ in Definition 5.1 by $S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)$ according to (5.87), (5.88) and by the Sobolev spaces $S^1_p W(\mathbb{Q}^n)$. For sake of clarity and also to show the dependence on the dimension $n \in \mathbb{N}$, $n \geq 2$, we write occasionally $\log^{n-1} l = (\log l)^{n-1}$, also in the version $(\log^{n-1} l)^\varkappa = (\log l)^{\varkappa(n-1)}$ where $\varkappa > 0$.

Theorem 5.15. *Let $\mathbb{Q}^n$ be the unit cube (5.84) in $\mathbb{R}^n$, $n \geq 2$.*

(i) *Let*

$$0 < p, q \leq \infty, \quad \frac{1}{p} < r < 1 + \min\left(\frac{1}{p}, 1\right). \tag{5.89}$$

Then

$$
\begin{aligned}
c_1\, l^{-r+(\frac{1}{p}-1)_+} (\log^{n-1} l)^{(1-\frac{1}{q})_+} &\le \mathrm{Int}_l\big(S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)\big) \le g^{\mathrm{lin}}_l\big(S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)\big)\\
&\le c_2\Big(\frac{l}{\log^{n-1} l}\Big)^{-r+(\frac{1}{p}-1)_+} (\log^{n-1} l)^{(1-\frac{1}{q})_+}
\end{aligned}
\tag{5.90}
$$

for some $c_1 > 0$, $c_2 > 0$ *and all* $l \in \mathbb{N}$, $l \ge 2$.

(ii) *Let* $1 < p < \infty$. *Then*

$$
c_1\, l^{-1}(\log l)^{\frac{n-1}{2}} \le \mathrm{Int}_l\big(S^1_p W(\mathbb{Q}^n)\big) \le g^{\mathrm{lin}}_l\big(S^1_p W(\mathbb{Q}^n)\big) \le c_2\, l^{-1}\, (\log l)^{3\frac{n-1}{2}} \tag{5.91}
$$

for some $c_1 > 0$, $c_2 > 0$ *and all* $l \in \mathbb{N}$, $l \ge 2$.

Proof. We outline the modifications compared with the previous two-dimensional considerations.

Step 1. We split $f \in S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)$ given by (5.86) with (5.88) as in (5.68),

$$
\begin{aligned}
f &= \sum_{k_1+\cdots+k_n\le K}\ \sum_{m\in\mathbb{P}^{F,n}_k} d^2_{km}(f)^n v_{km} + \sum_{k_1+\cdots+k_n> K}\ \sum_{m\in\mathbb{P}^{F,n}_k} d^2_{km}(f)^n v_{km}\\
&= M^K f + R^K f, \quad K \in \mathbb{N}.
\end{aligned}
\tag{5.92}
$$

First we estimate the remainder term $R^K f$ in $L_1(\mathbb{Q}^n)$. Instead of L in (4.138) and the subsequent estimates one has now L^{n-1}, compared with K by

$$
L^{n-1} = (L-K+K)^{n-1} = \sum_{t=0}^{n-1} c_t (L-K)^t K^{n-1-t}. \tag{5.93}
$$

This shows that one has now $2^{-Kr} K^{(n-1)(\frac{1}{p}-\frac{1}{q})}$ in place of $2^{-Kr} K^{\frac{1}{p}-\frac{1}{q}}$ in (4.138) and the subsequent estimates. The counterparts of (4.142), (4.144) with $u = 1$, are given now by

$$
\|R^K f\,|L_1(\mathbb{Q}^n)\| \le c\, 2^{-Kr+K(\frac{1}{p}-1)_+}\, K^{(n-1)(1-\frac{1}{q})_+} \|f\,|S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)\|, \quad K \in \mathbb{N}. \tag{5.94}
$$

One needs for $M^K f$ in (5.92) the knowledge of f in $K^{n-1}2^K$ points in $\overline{\mathbb{Q}^n}$. For given $l \in \mathbb{N}$, $l \ge 2$,

$$
K = \log l - \log(\log^{n-1} l) + \varepsilon, \quad 2^K \sim \frac{l}{\log^{n-1} l}, \quad K^{n-1}\, 2^K \sim l, \tag{5.95}
$$

is the n-dimensional version of (4.145). Then the right-hand side of (5.90) follows from (5.94), (5.95). Recall that we have always (5.8). As far as (5.90) is concerned it remains to prove the left-hand side.

Step 2. As for the left-hand side of (5.90) one can argue as in the Steps 3 and 4 in the proof of Theorem 4.15 with $u = 1$ using again that all functions involved are non-negative, in particular the counterparts of (4.149), (4.153),

$$f = \sum_{m \in \mathbb{P}_k^\Gamma} v_{km} \quad \text{and} \quad f = \sum_{k_1+\cdots+k_n=l+1} \sum_{m \in \mathbb{P}_k^\Gamma} v_{km}. \tag{5.96}$$

Then one obtains the left-hand sides of (5.90) by the same arguments as there.

Step 3. According to (3.196) one has (4.158), (4.159) with $\mathbb{Q}^n$ in place of $\mathbb{Q}^2$. In addition $S^1_2 W(\mathbb{Q}^n) = S^1_{2,2} B(\mathbb{Q}^n)$ $(= S^1_{2,2}\mathfrak{B}(\mathbb{Q}^n))$. Then (5.91) follows from (5.90) generalising (4.163) and (5.82). □

Remark 5.16. Let $S^1_1 W(\mathbb{R}^n)$ be the collection of all $f \in S'(\mathbb{R}^n)$ (or likewise $f \in L_1(\mathbb{R}^n)$) such that

$$\|f \,|S^1_1 W(\mathbb{R}^n)\| = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha_j \in \{0,1\}}} \|D^\alpha f \,|L_1(\mathbb{R}^n)\| < \infty, \tag{5.97}$$

and let $S^1_1 W(\mathbb{Q}^n)$ be its restriction to $\mathbb{Q}^n$. Then

$$\|f \,|S^1_1 W(\mathbb{Q}^n)\| = \sum_{\substack{\alpha \in \mathbb{N}_0^n \\ \alpha_j \in \{0,1\}}} \|D^\alpha f \,|L_1(\mathbb{Q}^n)\| \tag{5.98}$$

is an equivalent norm in $S^1_1 W(\mathbb{Q}^n)$. This can be shown as in the proof of Theorem 1.67. In particular smooth functions are dense both in $S^1_1 W(\mathbb{R}^n)$ and in $S^1_1 W(\mathbb{Q}^n)$. Let $x = (x', x_n) \in \mathbb{Q}^n$. Iterative application of

$$|f(x', x_n)| \le c \int_0^1 |f(x', y_n)| \, \mathrm{d}y_n + c \int_0^1 \left| \frac{\partial f}{\partial y_n}(x', y_n) \right| \mathrm{d}y_n \tag{5.99}$$

results in

$$|f(x)| \le c \, \|f \,|S^1_1 W(\mathbb{Q}^n)\|, \quad x \in \mathbb{Q}^n. \tag{5.100}$$

In particular,

$$S^1_p W(\mathbb{Q}^n) \hookrightarrow S^1_1 W(\mathbb{Q}^n) \hookrightarrow C(\mathbb{Q}^n), \quad 1 < p < \infty. \tag{5.101}$$

Hence one can apply Definition 5.1 to $G(\Omega) = S^1_1 W(\mathbb{Q}^n)$. Then one has by (5.91) and (5.101) that

$$c \, l^{-1} \, (\log l)^{\frac{n-1}{2}} \le \mathrm{Int}_l \big(S^1_1 W(\mathbb{Q}^n) \big) \tag{5.102}$$

for some $c > 0$ and all $l \in \mathbb{N}$, $l \ge 2$. This complements the left-hand side of (5.91). But it is not so clear by our approach whether the right-hand side of (5.91) can also be extended to $p = 1$. We glance at $n = 1$, hence $W^1_1(I)$ where $I = (0,1)$ is the unit interval. By (5.99) we have

$$W^1_p(I) \hookrightarrow W^1_1(I) \hookrightarrow C(I), \quad 1 < p < \infty. \tag{5.103}$$

Furthermore,

$$\operatorname{Int}_l\big(W_1^1(I)\big) \sim l^{-1}, \quad l \in \mathbb{N}. \tag{5.104}$$

The estimate from below follows from (5.33) with $A^s_{pq}(I) = W^1_p(I)$ and (5.103). As for the estimate from above we use an argument which is typical for our later considerations (in higher dimensions) about discrepancy. Let χ^l_m be the characteristic function of the interval $\big[\frac{m}{l}, 1\big]$ with $m = 0, \ldots, l-1$. Then the estimate from above follows from

$$\begin{aligned}\Big|\int_I f(x)\,\mathrm{d}x - l^{-1}\sum_{m=0}^{l-1} f\Big(\frac{m}{l}\Big)\Big| &= \Big|\int_I \Big(x - l^{-1}\sum_{m=0}^{l-1}\chi^l_m(x)\Big) f'(x)\,\mathrm{d}x\Big| \\ &\le l^{-1}\int_I |f'(x)|\,\mathrm{d}x.\end{aligned} \tag{5.105}$$

Hence the dominating order l^{-1} in (5.102) and (5.104) is the same. But also for $p = 1$ one has indispensable log-terms in higher dimensions.

Remark 5.17. It is well known that the left-hand side of (5.91) is the exact bound. This is a consequence of related assertions in the L_p-discrepancy theory which will be discussed in greater detail in Chapter 6 below. At this moment we only mention that in the context of an L_2-discrepancy theory the estimate of the integral numbers from below in

$$\operatorname{Int}_l\big(S^1_2 W(\mathbb{Q}^n)\big) \sim l^{-1}(\log l)^{\frac{n-1}{2}}, \quad l \in \mathbb{N},\ l \ge 2, \tag{5.106}$$

is a celebrated theorem by K. F. Roth, [Roth54], complemented by [Chen85], [Chen87]. Corresponding estimates from above are proved in [Roth80], [Fro80]. The extension of this theory from $p = 2$ to $1 < p < \infty$ goes back to [Schw77], [Tem90] (estimates from below) and [Chen80] (estimates from above), hence

$$\operatorname{Int}_l\big(S^1_p W(\mathbb{Q}^n)\big) \sim l^{-1}(\log l)^{\frac{n-1}{2}}, \quad l \in \mathbb{N},\ l \ge 2. \tag{5.107}$$

As said we return in Chapter 6 in greater detail to discrepancy and its relation to integration. Then we give also further references. Our approach to the lower bounds in (5.90), (5.91), and hence in (5.106), (5.107), is different. We rely on extremal functions of type (4.149), (4.153) and (5.96). This gives the exact lower bounds in (5.106), (5.107). As for the upper bounds in (5.107) there is little hope that our method can be improved such that one obtains (5.107). It seems to be a sophisticated art to find irregularly distributed points in $\mathbb{Q}^n$ producing in the context of the L_p-discrepancy theory, $1 < p < \infty$, exact upper bounds. We refer to [Roth80] and the explicit (but rather involved) constructions in [ChS02], [Skr06]. Our distribution of points which is related to the Smolyak algorithm and the so-called hyperbolic cross is too regular. This is also supported by [Pla00] dealing with the tractability (tight control of constants in dependence on the dimension $n \in \mathbb{N}$) in connection with numerical integration and discrepancy. It comes out that sparse grids, covering Smolyak algorithms and the distribution of points as used in our approach are not optimal. However encouraged by the above considerations one may ask whether the lower bound in (5.90) is the exact bound. In other words it would be of interest *to clarify whether*

$$\operatorname{Int}_l\big(S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)\big) \sim l^{-r+(\frac{1}{p}-1)_+}(\log^{n-1} l)^{(1-\frac{1}{q})_+}, \quad l \in \mathbb{N},\ l \ge 2, \quad (5.108)$$

with $0 < p, q \le \infty$, $\frac{1}{p} < r < 1 + \min(\frac{1}{p}, 1)$.

Using (5.107) one can confirm (5.108) in some cases:

If $2 \le p < \infty$ *then*

$$\operatorname{Int}_l\big(S^1_{p,2}B(\mathbb{Q}^n)\big) \sim l^{-1}(\log l)^{\frac{n-1}{2}}, \quad l \in \mathbb{N},\ l \ge 2. \quad (5.109)$$

This follows from (5.107), (4.159) (with $\mathbb{Q}^n$ in place of $\mathbb{Q}^2$) and the lower bound in (5.90) (recall that in this case $S^1_{p,2}B(\mathbb{Q}^n) = S^1_{p,2}\mathfrak{B}(\mathbb{Q}^n)$). We return to this problem later on in Remarks 6.25, 6.28 in the context of discrepancy numbers.

Remark 5.18. Another case of interest are the Hölder spaces $S^r\mathcal{C}(\mathbb{Q}^n) = S^r_{\infty\infty}B(\mathbb{Q}^n)$, $0 < r < 1$. They can be characterised by first differences (4.218) (where $n = 2$). It follows from (5.90) that

$$c_1\, l^{-r}(\log l)^{n-1} \le \operatorname{Int}_l\big(S^r\mathcal{C}(\mathbb{Q}^n)\big) \le c_2\, l^{-r}(\log l)^{(n-1)(r+1)} \quad (5.110)$$

for some $c_1 > 0$, $c_2 > 0$ and all $l \in \mathbb{N}$, $l \ge 2$. Similarly one can define spaces $S^rC(\mathbb{Q}^n)$ with $r \in \mathbb{N}$ as the collection of all functions $f \in C(\mathbb{Q}^n)$ having all classical derivatives $D^\alpha f \in C(\mathbb{Q}^n)$ of order $\alpha = (\alpha_1, \dots, \alpha_n)$ with $0 \le \alpha_j \le r$. Then one has the same upper bounds as in (5.110) now with $r \in \mathbb{N}$. We refer to [NoR96], [NoR99].

Remark 5.19. Numerical integration, especially in $\mathbb{Q}^n$, discrepancy and tractability have been considered in numerous papers and books. We refer to [NSTW09] as an introduction aimed at a wider audience, to [Woz09] which is a historical survey, and to [NoW08], [NoW09] for detailed studies. Quite often one prefers to deal with periodic functions in the n-torus $\mathbb{T}^n$ in place of $\mathbb{Q}^n$. We refer in particular to [Tem93]. The methods in the above-mentioned papers and books are different. Our approach, based on Faber bases, was first presented in [Tri09].

5.3.3 Integration based on logarithmic spaces

The logarithmic gaps between the left-hand sides and the right-hand sides in Theorem 5.13 for integral numbers are more or less special cases of corresponding logarithmic gaps for sampling numbers in Theorem 4.15. This can be extended to higher dimensions. We refer to Theorem 5.15 as far as integral numbers in higher dimensions are concerned. In Section 4.4.1 we took this situation, but also related assertions with respect to the unit interval $I = (0, 1)$, as a motivation to deal with sampling based on logarithmic spaces. The replacement of $S^r_{pq}\mathfrak{B}(\mathbb{Q}^2)$ in Theorem 4.15 by $S^{r,b}_{pp}\mathfrak{B}(\mathbb{Q}^2)$ in

Theorem 4.24 and Corollary 4.25 modifies the corresponding sampling numbers somewhat but preserves the logarithmic gap. In Definition 4.27 we introduced the spaces $S^{r,(b)}_{pq}B(\mathbb{Q}^2)$ reflecting a (logarithmically) different behaviour in $\mathbb{Q}^2$ and on its boundary $\partial\mathbb{Q}^2$. Then one has Theorem 4.29 and the satisfactory Corollary 4.30. Maybe their n-dimensional counterparts on $\mathbb{Q}^n$ in (4.281) and (4.284) are even more interesting. One can modify these considerations and proposals in the context of integral numbers. We restrict ourselves to the two-dimensional case. Let again

$$\mathbb{Q}^2 = \big\{x = (x_1, x_2) : 0 < x_1 < 1,\ 0 < x_2 < 1\big\} \tag{5.111}$$

be the unit square in $\mathbb{R}^2$ and let $S^{r,(b)}_{pq}B(\mathbb{Q}^2)$ be the spaces according to Definition 4.27. Let $g^{\mathrm{lin}}_k\big(G(\Omega)\big)$ and $\mathrm{Int}_k\big(G(\Omega)\big)$ now with $G(\Omega) = S^{r,(b)}_{pq}B(\mathbb{Q}^2)$ be the numbers introduced in Definition 5.1. Let

$$K_l = \log l - \log\log l + \varepsilon, \quad L_l = \log l + \delta, \quad 0 \le \varepsilon, \delta < 1,\ l \in \mathbb{N},\ l \ge 2, \tag{5.112}$$

be as in (4.263). We split the explicit formula (5.67) according to (4.264), hence

$$\begin{aligned}
\overline{M}_l f &= \int_{\mathbb{Q}^2} (M_l f)(x)\,\mathrm{d}x \\
&= \sum_{m_1,m_2\in\{0,1\}} 2^{-2} f(m_1,m_2) \\
&\quad - \sum_{k_2=0}^{L_l+1}\sum_{m_2=0}^{2^{k_2}-1}\sum_{m_1=0}^{1} 2^{-k_2-3}\Delta^2_{2^{-k_2-1},2} f\big(m_1, 2^{-k_2}m_2\big) \\
&\quad - \sum_{k_1=0}^{L_l+1}\sum_{m_1=0}^{2^{k_1}-1}\sum_{m_2=0}^{1} 2^{-k_1-3}\Delta^2_{2^{-k_1-1},1} f\big(2^{-k_1}m_1, m_2\big) \\
&\quad + \sum_{\substack{k_1+k_2\le K_l\\ k\in\mathbb{N}_0^2}}\sum_{m_1=0}^{2^{k_1}-1}\sum_{m_2=0}^{2^{k_2}-1} 2^{-k_1-k_2-4}\Delta^{2,2}_{2^{-k_1-1},2^{-k_2-1}} f\big(2^{-k_1}m_1, 2^{-k_2}m_2\big).
\end{aligned} \tag{5.113}$$

This evaluates $f \in C(\mathbb{Q}^2)$ at $\sim l$ points in $\overline{\mathbb{Q}^2}$.

Theorem 5.20. *Let*

$$1 \le p \le \infty, \quad \frac{1}{p} < r < 1 + \frac{1}{p}. \tag{5.114}$$

Then the embedding

$$\mathrm{id}\colon S^{r,(r+1-\frac{1}{p})}_{pp}B(\mathbb{Q}^2) \hookrightarrow L_1(\mathbb{Q}^2) \tag{5.115}$$

is compact and for some $c > 0$, *all* $l \in \mathbb{N}$, $l \ge 2$, *and all* $f \in S^{r,(r+1-\frac{1}{p})}_{pp}B(\mathbb{Q}^2)$,

$$\Big| \int_{\mathbb{Q}^2} f(x)\,\mathrm{d}x - \overline{M}_l f \Big| \le c\, l^{-r}\, \big\| f \,|S^{r,(r+1-\frac{1}{p})}_{pp}B(\mathbb{Q}^2)\big\|. \tag{5.116}$$

Furthermore,

$$\mathrm{Int}_l\big(S^{r,(r+1-\frac{1}{p})}_{pp}B(\mathbb{Q}^2)\big) \sim g^{\mathrm{lin}}_l\big(S^{r,(r+1-\frac{1}{p})}_{pp}B(\mathbb{Q}^2)\big) \sim l^{-r} \tag{5.117}$$

for all $l \in \mathbb{N}$.

Proof. By the proofs of Theorem 4.29 and Corollary 4.30 it follows that

$$\begin{aligned}\big\| f - M_l f \,|L_1(\mathbb{Q}^2)\big\| &\le \big\| f - M_l f \,|L_p(\mathbb{Q}^2)\big\| \\ &\le c\, l^{-r}\,\big\| f\,|S^{r,(r+1-\frac{1}{p})}_{pp}B(\mathbb{Q}^2)\big\|.\end{aligned} \tag{5.118}$$

This covers (5.116) and by (5.8) the estimate of the integral numbers from above in (5.117). Using that f in (4.261) is non-negative one can modify (4.262) by

$$\| f\,|S^{r,(r+1-\frac{1}{p})}_{pp}B(\mathbb{Q}^2)\| \sim 2^{lr}, \qquad \int_{\mathbb{Q}^2} f(x)\,\mathrm{d}x \sim 1. \tag{5.119}$$

This gives the estimate of the integral numbers from below in (5.117) with 2^l in place of l. □

Remark 5.21. According to Theorem 1.67 the spaces $S^r_{pp}B(\mathbb{Q}^2)$ with $1 \le p \le \infty$ and $r > 0$ can be equivalently normed by (1.294). We took for granted that this assertion can be extended to the spaces $S^{r,b}_{pp}B(\mathbb{Q}^2)$, equivalently normed by (1.369). In case of the spaces $S^{r,(b)}_{pp}B(\mathbb{Q}^2)$ with p, r as in (5.114) and $b \in \mathbb{R}$ the interior terms in (4.249) and the boundary terms are treated differently. As discussed in Remark 4.28 one has for the boundary term $f^{\partial\mathbb{Q}^2}$ in (4.252) that

$$\big\| f^{\partial\mathbb{Q}^2}\,|S^{r,(b)}_{pp}B(\mathbb{Q}^2)\big\| \sim \big\|\mathrm{tr}_{\partial\mathbb{Q}^2} f\,|B^r_{pp}(\partial\mathbb{Q}^2)\big\|, \tag{5.120}$$

where $B^r_{pp}(\partial\mathbb{Q}^2)$ are the spaces introduced in (3.121). Let p, r be as in (5.114) and let $b = r + 1 - \frac{1}{p}$. Then it follows from the above considerations that the spaces $S^{r,(b)}_{pp}B(\mathbb{Q}^2)$ can be equivalently normed by

$$\begin{aligned}&\big\| f\,|S^{r,(b)}_{pp}B(\mathbb{Q}^2)\big\|^* \\ &= \big\|\mathrm{tr}_{\partial\mathbb{Q}^2} f\,|B^r_{pp}(\partial\mathbb{Q}^2)\big\| \\ &\quad + \Big(\int_0^{1/2}\!\!\int_0^{1/2} (h_1h_2)^{-rp}|\log h_1|^{bp}|\log h_2|^{bp}\big\|\Delta^{2,2}_{h,\mathbb{Q}^2} f\,|L_p(\mathbb{Q}^2)\big\|^p \frac{\mathrm{d}h_1\mathrm{d}h_2}{h_1h_2}\Big)^{1/p}.\end{aligned} \tag{5.121}$$

As for $\Delta^{2,2}_{h,\mathbb{Q}^2} f$ we refer to (1.293). This is the continuous version of (4.249) with $p = q$. Recall that $S^1_2W(\mathbb{Q}^2) = S^1_{2,2}B(\mathbb{Q}^2)$ and that by (5.120) and Theorem 1.67,

$$\begin{aligned}\| f\,|S^1_2W(\mathbb{Q}^2)\| &\sim \big\|\mathrm{tr}_{\partial\mathbb{Q}^2} f\,|W^1_2(\partial\mathbb{Q}^2)\big\| \\ &\quad + \Big(\int_0^{1/2}\!\!\int_0^{1/2} (h_1h_2)^{-2}\big\|\Delta^{2,2}_{h,\mathbb{Q}^2} f\,|L_2(\mathbb{Q}^2)\big\|^2 \frac{\mathrm{d}h_1\mathrm{d}h_2}{h_1h_2}\Big)^{1/2}\end{aligned} \tag{5.122}$$

are equivalent norms. On the one hand we have (5.82), (5.83) for the integral numbers of $S^1_2W(\mathbb{Q}^2)$. On the other hand we obtained by (5.117) that

$$\mathrm{Int}_l\big(S^{1,(3/2)}_{2,2}B(\mathbb{Q}^2)\big) \sim l^{-1}, \quad l \in \mathbb{N}. \tag{5.123}$$

By (5.121) this logarithmic perturbation of $S^1_2W(\mathbb{Q}^2)$ can be normed by

$$\begin{aligned} &\big\| f \,|S^{1,(3/2)}_{2,2}B(\mathbb{Q}^2)\big\| \\ &= \big\| \mathrm{tr}_{\partial\mathbb{Q}^2} f \,|W^1_2(\partial\mathbb{Q}^2)\big\| \\ &\quad + \Big(\int_0^{1/2} \int_0^{1/2} (h_1h_2)^{-2} |\log h_1|^3 |\log h_2|^3 \big\| \Delta^{2,2}_{h,\mathbb{Q}^2} f \,|L_2(\mathbb{Q}^2)\big\|^2 \frac{\mathrm{d}h_1\mathrm{d}h_2}{h_1h_2}\Big)^{1/2}. \end{aligned} \tag{5.124}$$

In other words, the reinforcement of the interior part of $S^1_2W(\mathbb{Q}^2)$ in (5.124) by $|\log h_1|^3 |\log h_2|^3$ improves (5.82) by (5.123) and also (5.83) by (5.116). Our constructions our closely related to so-called Smolyak algorithms and to the hyperbolic cross (hyperbolic points). As discussed in Remark 5.17 there is little hope that these distributions of points in which $f(x)$ is evaluated give for the spaces considered in Theorem 5.13 similar results as for their logarithmic perturbations. The situation is similar as for sampling numbers where Corollary 4.30 is the counterpart of Theorem 5.20. But even in this context there arise several questions which we discussed in Point 4 in Section 4.4.4. In particular one may ask whether $b = r + 1 - \frac{1}{p}$ in Theorem 5.20 is the best possible choice.

Remark 5.22. We restricted the above considerations to the two-dimensional case, hence $\mathbb{Q}^2$. We outlined in Section 4.4.4 some proposals how to proceed in higher dimensions. In particular we described in (4.281) and (4.284) norms in suitable spaces which may be considered as n-dimensional generalisations of the spaces used in Theorem 5.20. Instead of sampling numbers one may now use these spaces in the context of numerical integration, including tractability, based on (4.284).

5.4 Integration in planar domains

5.4.1 Introduction, definitions

Recall that we introduced in Definition 5.1 the numbers $g^{\mathrm{lin}}_l\big(G(\Omega)\big)$ and $\mathrm{Int}_l\big(G(\Omega)\big)$ where Ω is a bounded domain in $\mathbb{R}^n$ and $G(\Omega)$ is a function space satisfying (5.2). Let $\Omega = I = (0,1)$ be the unit interval. Let

$$1 \le p \le \infty, \quad \frac{1}{p} < r < 1 + \frac{1}{p}. \tag{5.125}$$

Then

$$g^{\mathrm{lin}}_l\big(B^r_{pp}(I)\big) \sim \mathrm{Int}_l\big(B^r_{pp}(I)\big) \sim l^{-r}, \quad l \in \mathbb{N}, \tag{5.126}$$

is a special case of Theorem 5.7. We took this as a guide asking for the same behaviour in higher dimensions. By Theorem 5.4 one has

$$g_l^{\mathrm{lin}}\big(B^{rn}_{pp}(\Omega)\big) \sim \mathrm{Int}_l\big(B^{rn}_{pp}(\Omega)\big) \sim l^{-r}, \quad l \in \mathbb{N}, \tag{5.127}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, and p, r as in (5.125). It is somewhat surprising that one has (almost) the same behaviour for much larger spaces of (dominating mixed) smoothness r with p, r as in (5.125), in contrast to the smoothness rn for the isotropic spaces in (5.127). But one has to pay a price. Spaces with dominating mixed smoothness on $\mathbb{R}^n$ depend on fixed Euclidean coordinates. According to Definitions 1.56 and 1.81 one can introduce corresponding spaces on arbitrary domains Ω in $\mathbb{R}^n$ by restriction. But one obtains substantial assertions only for rectangles in $\mathbb{R}^n$ having sides parallel to the axes of coordinates with the unit cube $\mathbb{Q}^n$ in (5.84) as prototype. The first step beyond this philosophy was done in Definition 4.27 where we introduced the spaces $S^{r,(b)}_{pq}B(\mathbb{Q}^2)$ which are not obtained by restriction of corresponding spaces on $\mathbb{R}^2$. The behaviour of functions belonging to these spaces inside of $\mathbb{Q}^2$ and at the boundary $\partial\mathbb{Q}^2$ may be different, (5.120). We described in Section 4.4.4 possible n-dimensional generalisations. In any case one can use this decoupling of smoothness properties in $\mathbb{Q}^2$ and at $\partial\mathbb{Q}^2$ as a motivation to introduce logarithmic spaces of type $S^{r,(b)}_{pq}B$ also in more general domains. Of special interest might be corresponding spaces on the unit ball $\mathbb{B}^n$ and on the unit sphere $\mathbb{S}^n$,

$$\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}, \quad \mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}, \tag{5.128}$$

$n \geq 2$. Later on we restrict ourselves again to $n = 2$. But first we introduce a few definitions in $\mathbb{R}^n$, $n \geq 2$.

Let Ω be an arbitrary domain in $\mathbb{R}^n$ and let $u \in \mathbb{N}_0$. Then $C^u(\Omega)$ is the collection of all complex-valued functions f having classical derivatives up to order u inclusively in Ω such that any function $D^\alpha f$ with $|\alpha| \leq u$ can be extended continuously to $\overline{\Omega}$ and

$$\|f \,|C^u(\Omega)\| = \sum_{|\alpha|\leq u} \sup_{x\in\Omega} |D^\alpha f(x)| < \infty. \tag{5.129}$$

Furthermore, $C(\Omega) = C^0(\Omega)$ and $C^\infty(\Omega) = \bigcap_{u=0}^\infty C^u(\Omega)$. This coincides with [T08, Definition 5.17, p. 147] and goes back to [HaT08, Definition A.1, p. 246]. (It differs from other definitions in literature).

Recall that a one-to-one map ψ from a bounded domain Ω in $\mathbb{R}^n$ onto a bounded domain ω in $\mathbb{R}^n$,

$$\psi : \Omega \ni x \mapsto y = \psi(x) \in \omega, \tag{5.130}$$

is called a diffeomorphic map if

$$\psi_m \in C^\infty(\Omega) \quad \text{and} \quad (\psi^{-1})_m \in C^\infty(\omega), \quad m = 1, \ldots, n, \tag{5.131}$$

for the components ψ_m of ψ and $(\psi^{-1})_m$ of its inverse ψ^{-1},

$$\psi^{-1} \circ \psi = \mathrm{id} \text{ in } \Omega \quad \text{and} \quad \psi \circ \psi^{-1} = \mathrm{id} \text{ in } \omega. \tag{5.132}$$

A bounded domain Ω in $\mathbb{R}^n$ is said to be *diffeomorphic* to a bounded domain ω in $\mathbb{R}^n$ if there is a diffeomorphic map ψ of a neighbourhood of $\overline{\Omega}$ onto a neighbourhood of $\overline{\omega}$ with $\omega = \psi(\Omega)$. If Λ is a set in $\mathbb{R}^n$, then $\mathring{\Lambda}$ is the largest open set in $\mathbb{R}^n$ with $\mathring{\Lambda} \subset \Lambda$ (the interior of Λ). Let again

$$\mathbb{Q}^n = \big\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : 0 < x_m < 1\big\} \tag{5.133}$$

be the unit cube in $\mathbb{R}^n$ and let $\{0,1\}^n$ be its 2^n corner-points.

Definition 5.23. A domain Ω in $\mathbb{R}^n$, $n \ge 2$, is said to be *rectangular decomposable* if it is a bounded Lipschitz domain according to Definition 1.26 (ii) which can be represented as

$$\Omega = \Big(\bigcup_{l=1}^{L} \overline{\Omega_l}\Big)^{\circ} \quad \text{with } \Omega_l \cap \Omega_{l'} = \emptyset \text{ if } l \ne l', \tag{5.134}$$

such that each Ω_l is diffeomorphic to $\mathbb{Q}^n$, $\psi^l(\Omega_l) = \mathbb{Q}^n$ and $\overline{\Omega_l} \cap \overline{\Omega_{l'}}$ is either empty or a common face of Ω_l and $\Omega_{l'}$ of lower dimension, $l \ne l'$.

Remark 5.24. We are interested here in planar domains, Figure 5.1. The unit circle $\mathbb{B}^2$ can be decomposed as indicated. The annulus

$$\big\{x \in \mathbb{R}^2 : \tfrac{1}{2} < |x| < 1\big\} \tag{5.135}$$

can be furnished naturally with polar coordinates which may fit directly in what follows (but has not yet been done).

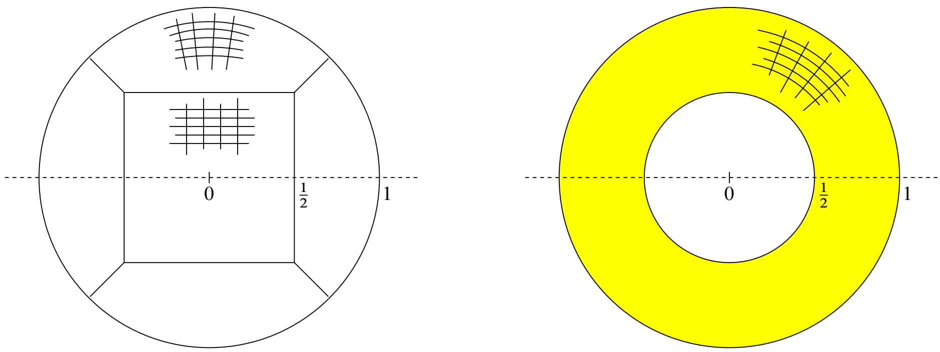

Figure 5.1. Planar domains.

5.4.2 Spaces with dominating mixed smoothness, integration

We are back to $n = 2$ and denote the unit square by

$$Q = \mathbb{Q}^2 = \big\{x = (x_1, x_2) \in \mathbb{R}^2,\ 0 < x_1 < 1,\ 0 < x_2 < 1\big\}. \tag{5.136}$$

Let $S^{r,(b)}_{pq}B(Q)$ be the spaces introduced in Definition 4.27.

Definition 5.25. Let Ω be a rectangular decomposable domain in $\mathbb{R}^2$ according to Definition 5.23 and let $\psi = \{\psi^l\}_{l=1}^L$ be the corresponding diffeomorphic maps of Ω_l onto Q, $\psi^l(\Omega_l) = Q$. Let

$$0 < p, q \le \infty, \quad \frac{1}{p} < r < 1 + \min\Big(\frac{1}{p}, 1\Big), \quad b \in \mathbb{R}. \tag{5.137}$$

Then $S^{r,(b)}_{pq}B(\Omega, \psi)$ is the collection of all $f \in C(\Omega)$ such that

$$\big(f|\Omega_l\big) \circ (\psi^l)^{-1} \in S^{r,(b)}_{pq}B(Q), \quad l = 1, \dots, L. \tag{5.138}$$

Furthermore,

$$\|f\,|S^{r,(b)}_{pq}B(\Omega,\psi)\| = \sum_{l=1}^{L} \|(f|\Omega_l) \circ (\psi^l)^{-1}|S^{r,(b)}_{pq}B(Q)\|. \tag{5.139}$$

Remark 5.26. We add a few comments. Recall that $C(\Omega) = C^0(\Omega)$ has the same meaning as in connection with (5.129). As before, $f|\Omega_l$ is the restriction of $f \in C(\Omega)$ to Ω_l. Then $g_l = (f|\Omega_l) \circ (\psi^l)^{-1} \in C(Q)$ and the above definition makes sense. By (4.248) one has

$$g_l(x) = \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(g_l)\, v_{km}(x), \quad x \in Q, \tag{5.140}$$

and (4.249) with g_l in place of f. It follows that

$$f(x) = \sum_{l=1}^{L} \sum_{k \in \mathbb{N}^2_{-1}} \sum_{m \in \mathbb{P}^F_k} d^2_{km}(g_l)\, \big(v_{km} \circ \psi^l\big)(x), \quad x \in \Omega. \tag{5.141}$$

There is a temptation to call $\{v_{km} \circ \psi^l\}$ a Faber system or a Faber basis in Ω and (5.141) a corresponding expansion. But some care is necessary. If $k \in \mathbb{N}^2_0$ then one has $v_{km}(x) = v_{k_1m_1}(x_1)\, v_{k_2m_2}(x_2)$ according to (5.59) based on (5.27). Then $v_{km} \circ \psi^l$ is also a product of distorted hat-functions and an acceptable building block. The situation is different for the boundary terms with $k \in \mathbb{N}^2_{-1} \setminus \mathbb{N}^2_0$. But this can be repaired. Let $x^0 \in \Omega$ be an interior corner point of several rectangular domains Ω_l and let $k = (-1,-1)$. This corresponds to the first line in (5.59) and (5.61), hence $f(x^0)$ in all cases involved. One can sum up in (5.141) the corresponding terms. The outcome is again a distorted hat-function (Lipschitz function) in Ω. The situation is similar for boundary terms related to the second and third line in (5.59). On a common boundary line $\overline{\Omega_l} \cap \overline{\Omega_{l'}}$, $l \ne l'$, connecting two corner points of two adjacent rectangular domains Ω_l and $\Omega_{l'}$ the representation of f might be different (distorted Faber bases originating from $I = (0,1)$). Since $f \in C(\Omega)$ one can replace one by the other and argue as before clipping together terms with the same factors in front. One may also consult (3.97)–(3.100) where we discussed in detail the decomposition of f in an interior, boundary and corner part and (3.133), (3.134) for related extension procedures which remain valid also in the above context. Let $(v_{km} \circ \psi^l)_\Omega$ be these modified hat-functions. Then

$\{(v_{km} \circ \psi^l)_\Omega\}$ is also a conditional basis in $C(\Omega)$ and the coefficients evaluate f at distinguished points in $\overline{\Omega}$.

In other words, one can first deal with the corner points, then with the sides of $\overline{\Omega_l}$ and finally with the interior or Ω_l.

We refer in this context also to Remark 3.12. In particular, $C(\Omega)$ is isomorphic to $C(I)$ and any basis in $C(\Omega)$ is conditional. After these preparations one can now extend assertions for spaces $S^{r,(b)}_{pq}B$ from $\mathbb{Q}^2$ to rectangular decomposable domains Ω in $\mathbb{R}^2$. We restrict ourselves to the counterpart of Theorem 5.20. Let $g^{\text{lin}}_l\big(G(\Omega)\big)$ and $\text{Int}_l\big(G(\Omega)\big)$ be the numbers as introduced in Definition 5.1.

Theorem 5.27. *Let Ω be a rectangular decomposable domain in $\mathbb{R}^2$ according to Definition 5.23 and let $S^{r,(r+1-\frac{1}{p})}_{pp}B(\Omega,\psi)$ with*

$$1 \le p \le \infty, \quad \frac{1}{p} < r < 1 + \frac{1}{p}, \tag{5.142}$$

be the spaces as introduced in Definition 5.25. Then

$$\text{id}\colon S^{r,(r+1-\frac{1}{p})}_{pp}B(\Omega,\psi) \hookrightarrow L_1(\Omega) \tag{5.143}$$

is compact. Furthermore,

$$\text{Int}_l\big(S^{r,(r+1-\frac{1}{p})}_{pp}B(\Omega,\psi)\big) \sim g^{\text{lin}}_l\big(S^{r,(r+1-\frac{1}{p})}_{pp}B(\Omega,\psi)\big) \sim l^{-r} \tag{5.144}$$

for all $l \in \mathbb{N}$.

Proof. The estimates from below are local and essentially covered by (5.119), transferred to one of the rectangular domains Ω_l. As for estimates from above one can rely directly on (5.141) and previous arguments. But one can also transfer this question to $Q = \mathbb{Q}^2$. Then the diffeomorphic map $y = \psi^l(x)$ produces in the corresponding integral the Jacobian as an additional factor. But this is a real smooth function bounded from above and below by positive constants. This does not influence the previous arguments. □

Remark 5.28. The spaces $S^{r,(b)}_{pq}B(\Omega,\psi)$ look like patchwork quilts. The distinguished directions determining dominating mixed smoothness may change from one rectangular domain Ω_l to the next one, but only in moderate manner since they have common sides. Maybe the annulus (5.135) is an exception and it might well be reasonable to deal with spaces having dominating mixed smoothness in terms of polar coordinates directly without cutting this domain in pieces.

Chapter 6
Discrepancy

6.1 Introduction, definitions

6.1.1 Definitions

Discrepancy measures the deviation of sets of points from uniformity, preferably in the unit cube

$$\mathbb{Q}^n = \big\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : 0 < x_l < 1;\ l = 1, \ldots, n \big\} \tag{6.1}$$

in $\mathbb{R}^n$. This theory goes back to H. Weyl, [Weyl16], and J. G. van der Corput, [Cor35a], [Cor35b]. Nowadays it is a flourishing field of research with dozens of books and thousands of papers. A description of the work of Weyl may be found in several books. We refer in particular to [DiP10]. Van der Corput coined the word *Diskrepanz* (= discrepancy). Our aim is very restrictive. We deal with discrepancy mainly in the context of spaces with dominating mixed smoothness and numerical integration.

Let $S^r_{pq}A(\mathbb{Q}^n)$, $2 \le n \in \mathbb{N}$, be the spaces with dominating mixed smoothness as introduced in Definition 1.56. Let χ_R be the characteristic function of the rectangle

$$R = \big\{ x \in \mathbb{Q}^n : a_l < x_l < b_l;\ l = 1, \ldots, n \big\} \subset \mathbb{Q}^n \tag{6.2}$$

where $0 \le a_l < b_l \le 1$. Let $\Gamma = \{x^j\}_{j=1}^k \subset \mathbb{Q}^n$ be a set of k points in $\mathbb{Q}^n$. Then χ_{R^j} is the characteristic function of the rectangle

$$R^j = \big\{ x \in \mathbb{Q}^n : x^j_l < x_l < 1;\ l = 1, \ldots, n \big\}, \quad j = 1, \ldots, k, \tag{6.3}$$

anchored at the upper right corner of $\mathbb{Q}^n$ (and with x^j as the lower left corner). Let $A = \{a_j\}_{j=1}^k \subset \mathbb{C}$. The *discrepancy functions*

$$\operatorname{disc}_{\Gamma,A}(x) = \prod_{l=1}^n x_l - \sum_{j=1}^k a_j\, \chi_{R^j}(x), \quad x \in \mathbb{Q}^n, \tag{6.4}$$

and

$$\operatorname{disc}_{\Gamma}(x) = \prod_{l=1}^n x_l - \frac{1}{k} \sum_{j=1}^k \chi_{R^j}(x), \quad x \in \mathbb{Q}^n, \tag{6.5}$$

compare the volume of the rectangle with 0 as the lower left corner and x as the upper right corner with the weighted number of points $x^j \in \Gamma$ within this rectangle. The distinguished case $a_j = k^{-1}$ in (6.5) is of special interest. The above discrepancy functions are preferably measured in L_p-norms, $1 \le p \le \infty$, where nowadays

$p = 1$ and $p = \infty$ are the most interesting and most challenging cases. We wish to complement this theory measuring these discrepancy functions not only in $L_p(\mathbb{Q}^n)$ but also in suitable spaces $S^r_{pq}A(\mathbb{Q}^n)$. Let $\mathbb{Q}^1 = I = (0,1)$ be the unit interval in $\mathbb{R}$.

Definition 6.1. (i) Let $2 \le n \in \mathbb{N}$. Let $0 < p, q \le \infty$ ($p < \infty$ for F-spaces) and $r \in \mathbb{R}$ such that

$$\chi_R \in S^r_{pq}A(\mathbb{Q}^n) \quad \text{for any rectangle } R \tag{6.6}$$

in $\mathbb{Q}^n$ according to (6.2). Then

$$\mathrm{disc}_k\big(S^r_{pq}A(\mathbb{Q}^n)\big) = \inf \big\| \mathrm{disc}_{\Gamma,A} \,|S^r_{pq}A(\mathbb{Q}^n)\big\|, \quad k \in \mathbb{N}, \tag{6.7}$$

where the infimum is taken over all $\Gamma = \{x^j\}_{j=1}^k \subset \mathbb{Q}^n$ and all $A = \{a_j\}_{j=1}^k \subset \mathbb{C}$.

(ii) Let $n \in \mathbb{N}$ and $1 \le p \le \infty$. Then

$$\mathrm{disc}_k\big(L_p(\mathbb{Q}^n)\big) = \inf \big\| \mathrm{disc}_{\Gamma,A} \,|L_p(\mathbb{Q}^n)\big\|, \quad k \in \mathbb{N}, \tag{6.8}$$

where the infimum is taken over all $\Gamma = \{x^j\}_{j=1}^k \subset \mathbb{Q}^n$ and all $A = \{a_j\}_{j=1}^k \subset \mathbb{C}$.

(iii) Let $n \in \mathbb{N}$ and $1 \le p \le \infty$. Then

$$\mathrm{disc}^*_k\big(L_p(\mathbb{Q}^n)\big) = \inf \big\| \mathrm{disc}_{\Gamma} \,|L_p(\mathbb{Q}^n)\big\|, \quad k \in \mathbb{N}, \tag{6.9}$$

where the infimum is taken over all $\Gamma = \{x^j\}_{j=1}^k \subset \mathbb{Q}^n$.

Remark 6.2. We call $\mathrm{disc}_k\big(S^r_{pq}A(\mathbb{Q}^n)\big)$, $\mathrm{disc}_k\big(L_p(\mathbb{Q}^n)\big)$ and $\mathrm{disc}^*_k\big(L_p(\mathbb{Q}^n)\big)$ *discrepancy numbers*. One can incorporate $n = 1$ in part (i) with $A^r_{pq}(I)$ in place of $S^r_{pq}A(\mathbb{Q}^n)$. This will be done in some detail in Section 6.3.2. For this reason we assumed now $2 \le n \in \mathbb{N}$ in part (i). To provide a better understanding we discuss in the following Section 6.1.2 briefly the well-known assertion

$$\mathrm{disc}^*_k\big(L_p(I)\big) \sim k^{-1}, \quad k \in \mathbb{N}, \tag{6.10}$$

where $1 \le p \le \infty$ and again $I = (0,1)$ is the unit interval in $\mathbb{R}$. Let $n \ge 2$. After the above-mentioned papers by Weyl and van der Corput the next crucial step is due to K. F. Roth, [Roth54]. He proved that there is a constant $c > 0$ such that

$$\mathrm{disc}^*_k\big(L_2(\mathbb{Q}^n)\big) \ge c\, k^{-1} (\log k)^{\frac{n-1}{2}}, \quad 2 \le k \in \mathbb{N}. \tag{6.11}$$

It is this additional log-factor in comparison with (6.10) which indicates the unavoidable deviation from uniformity. A brief description of Roth's method may be found in [ChT09]. Basically he uses the orthogonal Haar tensor basis (2.315) in $L_2(\mathbb{Q}^n)$ and constructs extremal functions ensuring (6.11) in a similar way as in (5.96) based on (4.149), (4.153). Roth's method has been used later on in many other papers. A Fourier-analytical proof of (6.11) may be found in [BeC87] and [DrT97, Theorem 1.40, p. 29]. We refer also to [DiP10] for a different proof. As far as the discrepancies (6.8), (6.9) are concerned we give later on more detailed references. At this moment we only mention that further information about discrepancy, numerical integration and their mutual relations can be found in the books [BeC87], [DiP10], [DrT97], [Nie92], [Nov88], [NSTW09], [NoW08], [NoW09], [Tem93] and the survey [Tem03]. We stick in (6.4), (6.5) to the volume $\prod_{l=1}^n x_l$ of the rectangles anchored at the origin. But there are numerous variations dealing with rectangles anchored at other points in $\overline{\mathbb{Q}^n}$ or unanchored. Instead of rectangles also other sets and their measures have been considered. We refer to the above books. As far as probabilistic aspects and average case settings are concerned one may consult [Woz91], [NoW09].

In connection with part (i) of Definition 6.1 it is of interest to characterise the spaces $S^r_{pq}A(\mathbb{Q}^n)$ satisfying (6.6). Of course, spaces covered by Proposition 2.34 (and its n-dimensional generalisation) or having Haar tensor basis according to Section 2.4 satisfy (6.6). But the requirement (6.6) is much weaker than asking for Haar tensor bases. We outline a final answer.

Proposition 6.3. *Let χ_R be the characteristic function of the rectangle R according to* (6.2) *in $\mathbb{Q}^n$, $n \ge 2$.*

(i) *Let $0 < p, q \le \infty$, $r \in \mathbb{R}$. Then $\chi_R \in S^r_{pq}B(\mathbb{Q}^n)$ for any R with* (6.2) *if, and only if,*

$$\begin{cases} 0 < p \le \infty,\ 0 < q \le \infty, & r < 1/p, \\ 0 < p \le \infty,\ q = \infty, & r = 1/p. \end{cases} \tag{6.12}$$

(ii) *Let $0 < p < \infty$, $0 < q \le \infty$, $r \in \mathbb{R}$. Then $\chi_R \in S^r_{pq}F(\mathbb{Q}^n)$ for any R with* (6.2) *if, and only if,*

$$0 < p < \infty, \quad 0 < q \le \infty, \quad r < 1/p. \tag{6.13}$$

Proof. *Step* 1. It is sufficient to deal with $n = 2$. According to Definition 1.56 the spaces $S^r_{pq}A(\mathbb{Q}^2)$ are restrictions of $S^r_{pq}A(\mathbb{R}^2)$ to $\mathbb{Q}^2$. By dilation and translation arguments it is sufficient to clarify for which spaces one has

$$\chi \in S^r_{pq}A(\mathbb{R}^2), \quad \chi \text{ characteristic function of } \mathbb{Q}^2. \tag{6.14}$$

Recall that

$$\begin{aligned} &\|f \,|S^r_{pq}A(\mathbb{R}^2)\| \\ &\sim \|f\,|S^{r-1}_{pq}A(\mathbb{R}^2)\| + \sum_{l=1}^2 \Big\| \frac{\partial f}{\partial x_l} \,|S^{r-1}_{pq}A(\mathbb{R}^2)\Big\| + \Big\| \frac{\partial^2 f}{\partial x_1 \partial x_2} \,|S^{r-1}_{pq}A(\mathbb{R}^2)\Big\|, \end{aligned} \tag{6.15}$$

[ST87, Theorem 2, pp. 98/99]. In particular it follows from (6.14) that

$$\frac{\partial^2 \chi}{\partial x_1 \partial x_2} \in S^{r-1}_{pq}A(\mathbb{R}^2). \tag{6.16}$$

If $\varphi \in S(\mathbb{R}^2)$ then

$$\int_{\mathbb{R}^2} \frac{\partial^2 \chi}{\partial x_1 \partial x_2}(x)\, \varphi(x)\, \mathrm{d}x = \int_{\mathbb{Q}^2} \frac{\partial^2 \varphi}{\partial x_1 \partial x_2}(x)\, \mathrm{d}x = \varphi(1,1) - \varphi(1,0) - \varphi(0,1) + \varphi(0,0). \tag{6.17}$$

This shows that (6.16) is equivalent to

$$\delta \in S^{r-1}_{pq} A(\mathbb{R}^2). \tag{6.18}$$

Since $\hat{\delta} = b \neq 0$ is constant it follows from (1.151), based on (1.148), (1.6) that

$$\| \delta \, | S^{r-1}_{pq} B(\mathbb{R}^2) \| \geq c \Big(\sum_{k \in \mathbb{N}^2} 2^{(r-1)(k_1+k_2)q} \, 2^{(k_1+k_2)(1-\frac{1}{p})q} \Big)^{1/q} \tag{6.19}$$

for some $c > 0$ (modification if $q = \infty$). This shows that (6.12) is necessary. To prove that (6.13) is necessary for F-spaces one has to show that $\chi \notin S^{1/p}_{p\infty} F(\mathbb{R}^2)$. Using (1.153) with $r = \frac{1}{p}$, $q = \infty$, and the above arguments then one has to check whether

$$\sup_{k \in \mathbb{N}^2} \big| 2^{(k_1+k_2)(1+\frac{1}{p})} \varphi^\vee \big(2^{k_1} \xi_1 \big) \, \varphi^\vee \big(2^{k_2} \xi_2 \big) \big| \notin L_p(\mathbb{R}^2), \tag{6.20}$$

where $\varphi(t) = \varphi_0(t) - \varphi_0(2t)$, $t \in \mathbb{R}$, according to (1.6). We may assume that $\varphi(t) \geq 0$, $\varphi(t) = \varphi(-t)$ and $\varphi^\vee(\tau) \geq c > 0$ if, say, $|\tau| \leq 2$. Then

$$2^{(k_1+k_2)(1+\frac{1}{p})} \, \varphi^\vee \big(2^{k_1} \xi_1 \big) \, \varphi^\vee \big(2^{k_2} \xi_2 \big) \sim |\xi_1 \xi_2|^{-(1+\frac{1}{p})} \quad \text{if } |\xi_1| \sim 2^{-k_1}, \; |\xi_2| \sim 2^{-k_2}. \tag{6.21}$$

This proves (6.20). Hence (6.13) is necessary.

Step 2. It remains to prove (6.14) with p, q, r in (6.12) for B-spaces and with (6.13) for F-spaces. By elementary embedding it is sufficient to deal with $S^{1/p}_{p\infty} B(\mathbb{R}^2)$, $0 < p \leq \infty$. One can argue as above, relying on (6.15). Then one has (6.18) with $r = \frac{1}{p}$, $q = \infty$, $A = B$, and corresponding assertions for the three other terms on the right-hand side of (6.15). But this can also be done directly using the well-known (one-dimensional) Fourier transform of the characteristic function of the unit interval $I = (0, 1)$. □

Remark 6.4. Proposition 2.34 and elementary embeddings prove $\chi \in S^r_{pq} A(\mathbb{R}^2)$ in all cases covered by the above proposition with exception of the limiting case $S^{1/p}_{p,\infty} B(\mathbb{R}^2)$.

6.1.2 The one-dimensional case

In what follows in this Chapter 6 we deal mainly with discrepancies in $\mathbb{Q}^n$ where $2 \leq n \in \mathbb{N}$ as introduced in Definition 6.1. The one-dimensional case with the unit interval $\mathbb{Q}^1 = I = (0, 1)$ as underlying domain will be considered in Section 6.3.2. Then $S^r_{pq} A(\mathbb{Q}^1)$ must be identified with $A^r_{pq}(I)$. But to underline the substantial differences between one and higher dimensions we insert now an elementary proof of (6.10).

Proposition 6.5. *Let $\mathbb{Q}^1 = I = (0,1)$ be the unit interval in $\mathbb{R}$ and let $\operatorname{disc}_k^*\big(L_p(I)\big)$ be the discrepancy numbers according to (6.9), $1 \le p \le \infty$. Then there are two positive numbers c_1 and c_2 such that*

$$c_1\, k^{-1} \le \operatorname{disc}_k^*\big(L_1(I)\big) \le \operatorname{disc}_k^*\big(L_p(I)\big) \le \operatorname{disc}_k^*\big(L_\infty(I)\big) \le c_2\, k^{-1} \tag{6.22}$$

for $k \in \mathbb{N}$.

Proof. If $\Gamma = \{x^j\}_{j=1}^k \subset I$ with $0 < x^1 < \cdots < x^k < 1$ and $x^{j+1} - x^j \sim k^{-1}$ is inserted in (6.5) and (6.9) with $p = \infty$ then one obtains the right-hand side of (6.22). The monotonicity of $\operatorname{disc}_k^*\big(L_p(I)\big)$ with respect to p is obvious (Hölder's inequality). We prove the left-hand side of (6.22). Let $\Gamma = \{x^j\}_{j=1}^k \subset I$ with $0 < x^1 < x^2 < \cdots < x^k < 1$, complemented by $x^0 = 0$ and $x^{k+1} = 1$. Then

$$\int_0^1 \Big| x - \frac{1}{k} \sum_{j=1}^k \chi_{R^j}(x) \Big| \,\mathrm{d}x \ge c \sum_{l=0}^k (x^{l+1} - x^l)^2 \tag{6.23}$$

for some $c > 0$ (independent of Γ). Let $\{c_j\}_{j=1}^N \subset \mathbb{R}$ with $\sum_{j=1}^N c_j = 1$ where $N \in \mathbb{N}$. Then

$$\begin{aligned} \sum_{j=1}^N c_j^2 = \sum_{j=1}^N \Big(c_j - \frac{1}{N} + \frac{1}{N} \Big)^2 &= \sum_{j=1}^N \Big(c_j - \frac{1}{N} \Big)^2 + \frac{2}{N} \sum_{j=1}^N \Big(c_j - \frac{1}{N} \Big) + \frac{1}{N} \\ &= \sum_{j=1}^N \Big(c_j - \frac{1}{N} \Big)^2 + \frac{1}{N}. \end{aligned} \tag{6.24}$$

The left-hand side of (6.22) follows now from (6.23), (6.24). □

Remark 6.6. These simple arguments do not work in higher dimensions. Then we relate the discrepancy numbers in Definition 6.1 to suitable integral numbers according to (5.5). This can also be done in one dimension. We refer to (5.105) which might be considered as a one-dimensional forerunner of what follows. Otherwise we return to the one-dimensional case in greater detail in Section 6.3.2 below.

6.2 Relationships between integral and discrepancy numbers

6.2.1 Prerequisites

In Definitions 5.1 and 6.1 we introduced integral numbers and discrepancy numbers. It is the main aim of Section 6.2 to show that they are closely related. First we complement some previous assertions.

1. Integral numbers. Let $\mathbb{Q}^n$ be the unit cube (6.1) in $\mathbb{R}^n$, $n \ge 2$. In (5.86)–(5.88) we introduced the spaces

$$S^r_{pq}\mathfrak{B}(\mathbb{Q}^n), \quad 0 < p,q \le \infty, \ \tfrac{1}{p} < r < 1 + \min\big(\tfrac{1}{p}, 1\big), \tag{6.25}$$

in terms of the Faber system $\{v_{km}\}$ in (5.85), Figure 3.1, p. 127. One has by (4.123)–(4.125) and the comments in Section 4.4.4 that

$$S^r_{pq}\mathfrak{B}(\mathbb{Q}^n) = S^r_{pq}B(\mathbb{Q}^n) \ \text{ if } \begin{cases} p = q = \infty, & 0 < r < 1, \\ 0 < p,q < \infty, & \tfrac{1}{p} < r < 1 + \min\big(\tfrac{1}{p}, 1\big), \end{cases} \tag{6.26}$$

where $S^r_{pq}B(\mathbb{Q}^n)$ is the restriction of $S^r_{pq}B(\mathbb{R}^n)$ to $\mathbb{Q}^n$. Theorem 5.15 estimates the integral numbers $\mathrm{Int}_l\big(S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)\big)$ for these spaces.

2. Haar tensor bases. Relationships between integral numbers and discrepancy numbers are governed by an intense interplay between Faber bases and Haar tensor bases on $\mathbb{Q}^n$. Let

$$\big\{h_{km} : k \in \mathbb{N}^n_{-1}, \ m \in \mathbb{P}^{H,n}_k\big\} \tag{6.27}$$

be the orthogonal Haar tensor basis (2.315) in $\mathbb{Q}^n$ generalising (2.284)–(2.286) from $n = 2$ to $2 \le n \in \mathbb{N}$. According to Theorem 2.41 (i) and its indicated generalisation from $n = 2$ to $2 \le n \in \mathbb{N}$ in (2.318), (2.319) one can characterise suitable spaces $S^r_{pq}B(\mathbb{Q}^n)$ in terms of Haar tensor bases under natural conditions for p, q, r. But we had to exclude for technical reasons spaces with $p = \infty$, $0 < q \le 1$. This can be incorporated similarly as in (6.25), (6.26). Let $s^H_{pq}b(\mathbb{Q}^n)$ be the sequence spaces according to (2.316), (2.317). Then

$$S^r_{pq}\mathcal{B}(\mathbb{Q}^n), \quad 0 < p,q \le \infty, \ \tfrac{1}{p} - 1 < r < \min\big(\tfrac{1}{p}, 1\big), \tag{6.28}$$

is the collection of all $f \in D'(\mathbb{Q}^n)$ which can be represented as

$$f = \sum_{k \in \mathbb{N}^n_{-1}} \sum_{m \in \mathbb{P}^{H,n}_k} \lambda_{km}\, 2^{-(k_1 + \cdots + k_n)(r - \frac{1}{p})} h_{km}, \quad \lambda \in s^H_{pq}b(\mathbb{Q}^n). \tag{6.29}$$

This representation is unique with $\lambda_{km} = \lambda_{km}(f)$ as in (2.319). Let

$$\|f \,|S^r_{pq}\mathcal{B}(\mathbb{Q}^n)\| = \|\lambda(f)\,|s^H_{pq}b(\mathbb{Q}^n)\|. \tag{6.30}$$

One has by the n-dimensional version of Theorem 2.41 (i) that

$$S^r_{pq}\mathcal{B}(\mathbb{Q}^n) = S^r_{pq}B(\mathbb{Q}^n) \ \text{ if } \begin{cases} p = \infty, \ 1 < q \le \infty, & -1 < r < 0, \\ 0 < p < \infty, \ 0 < q \le \infty, & \tfrac{1}{p} - 1 < r < \min\big(\tfrac{1}{p}, 1\big). \end{cases} \tag{6.31}$$

In other words, we complemented the above spaces $S^r_{pq}B(\mathbb{Q}^n)$ by some limiting cases where $p = \infty$ and $q \le 1$. But it might well be the case that one has (6.31) also for these exceptional spaces.

3. Mappings. We extend the definition of $S^r_{pq}B(\mathbb{Q}^n)^\urcorner$ in (3.187) with p, q, r as in (3.155) = (3.192) to all spaces covered by (6.25),

$$S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)^\urcorner = \big\{ f \in S^r_{pq}\mathfrak{B}(\mathbb{Q}^n) : f|\partial\mathbb{Q}^n_1 = 0 \big\}. \tag{6.32}$$

This makes sense and causes no problems by the preceding considerations. It gives the possibility to extend Proposition 3.22 to these limiting cases:

Let

$$0 < p, q \le \infty, \quad \frac{1}{p} < r < 1 + \min\Big(\frac{1}{p}, 1\Big). \tag{6.33}$$

Then

$$\frac{\partial^n}{\partial x_1 \ldots \partial x_n} : f \mapsto \frac{\partial^n f}{\partial x_1 \ldots \partial x_n} \tag{6.34}$$

is an isomorphic map of $S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)^\urcorner$ *onto* $S^{r-1}_{pq}\mathcal{B}(\mathbb{Q}^n)$.

The complementing limiting spaces are defined in terms of Faber bases and Haar tensor bases. Then one can apply the n-dimensional version of (3.159), (3.160) also to these limiting spaces.

4. Duality. We need some duality assertions for the spaces in (6.28). Let

$$1 \le p, q < \infty, \quad \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1, \quad \frac{1}{p} - 1 < r < \frac{1}{p}. \tag{6.35}$$

Then one has in the context of the dual pairing $\big(D(\mathbb{Q}^n), D'(\mathbb{Q}^n)\big)$ that

$$S^r_{pq}B(\mathbb{Q}^n)' = S^{-r}_{p'q'}B(\mathbb{Q}^n). \tag{6.36}$$

All spaces are covered by (6.31). The duality (6.36) follows essentially from (2.272), (2.273) and the arguments in the proof of Theorem 2.41. But we add a few further comments. Let $s^H_{pq}b(\mathbb{Q}^n)$ be the sequence space according to (2.316), (2.317). Then one has

$$s^H_{pq}b(\mathbb{Q}^n)' = s^H_{p'q'}b(\mathbb{Q}^n), \quad 1 \le p, q < \infty, \ \tfrac{1}{p} + \tfrac{1}{p'} = \tfrac{1}{q} + \tfrac{1}{q'} = 1, \tag{6.37}$$

in the usual interpretation as the dual pairing

$$(\lambda, \mu) = \sum_{k \in \mathbb{N}^n_{-1}} \sum_{m \in \mathbb{P}^{H,n}_k} \lambda_{km}\, \mu_{km}, \quad \lambda \in s^H_{pq}b(\mathbb{Q}^n), \quad \mu \in s^H_{p'q'}b(\mathbb{Q}^n). \tag{6.38}$$

This can be proved by standard arguments. But it follows also from Section 1.11.1, p. 68 in [T78]. By Theorem 2.41 (i) and its n-dimensional version according to (2.318) the finite linear combinations of the Haar tensor functions h_{km} are dense in $S^r_{pq}B(\mathbb{Q}^n)$ with (6.35). Any function h_{km} can be approximated in $S^r_{pq}B(\mathbb{R}^n)$, and hence in $S^r_{pq}B(\mathbb{Q}^n)$, by functions belonging to $D(\mathbb{Q}^n)$ (recall that $g(\cdot + y) \to g$ in this space if $y \to 0$). Hence $D(\mathbb{Q}^n)$ is dense in $S^r_{pq}B(\mathbb{Q}^n)$ and the assertion (6.36) makes sense as a dual pairing within $\big(D(\mathbb{Q}^n), D'(\mathbb{Q}^n)\big)$. If $f \in S^r_{pq}B(\mathbb{Q}^n)$ is represented by (2.318) and similarly $g \in S^{-r}_{p'q'}B(\mathbb{Q}^n)$ by

$$g = \sum_{k\in\mathbb{N}^n_{-1}} \sum_{m\in\mathbb{P}^{H,n}_k} \mu_{km}\, 2^{-(k_1+\dots+k_n)(-r-\frac{1}{p'})} h_{km}, \quad \mu \in s^H_{p'q'}b(\mathbb{Q}^n), \tag{6.39}$$

then it follows from the orthogonality of h_{km} that

$$(f,g) = \sum_{k,m} \lambda_{km}\, \mu_{km}\, 2^{k_1+\dots+k_n} \int_{\mathbb{Q}^n} h^2_{km}(x)\,\mathrm{d}x = \sum_{k,m} \lambda_{km}\, \mu_{km}. \tag{6.40}$$

The duality (6.37) and the n-dimensional version of Theorem 2.41 (i) prove (6.36).

6.2.2 Hlawka–Zaremba identity

Let $\mathbb{Q}^n$ be again the cube (6.1) and let $S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)^{\urcorner}$ be the spaces according to (6.32), (6.33), subspaces of (6.25), (6.26). According to the n-dimensional version of (3.159) they can be represented as

$$f = \sum_{k\in\mathbb{N}^n_{-1}} \sum_{m\in\mathbb{P}^{H,n}_k} \lambda_{km}\, 2^{-(k_1+\dots+k_n)(r-\frac{1}{p})} v_{km}, \quad \lambda \in s^H_{pq}b(\mathbb{Q}^n), \tag{6.41}$$

by the reduced Faber basis

$$\big\{v_{km} : k \in \mathbb{N}^n_{-1},\ m \in \mathbb{P}^{H,n}_k\big\} \tag{6.42}$$

with the same index set as for the Haar tensor basis in (6.29). Compared with the full Faber system (5.85)=(3.178), based on the n-dimensional version of (3.60)–(3.62), the above reduced Faber system consists of all Faber functions v_{km} with

$$v_{km}|\partial\mathbb{Q}^n_1 = 0, \quad k \in \mathbb{N}^n_{-1},\ m \in \mathbb{P}^{H,n}_k. \tag{6.43}$$

By (3.183) with (3.182) one has always

$$S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)^{\urcorner} \hookrightarrow \big\{g \in C(\mathbb{Q}^n) : g|\partial\mathbb{Q}^n_1 = 0\big\}. \tag{6.44}$$

Theorem 6.7. *Let $\mathbb{Q}^n$ be the unit cube (6.1) and let $\mathrm{disc}_{\Gamma,A}$ be the discrepancy function (6.4). Then*

$$(-1)^n \int_{\mathbb{Q}^n} \mathrm{disc}_{\Gamma,A}(x)\, \frac{\partial^n f(x)}{\partial x_1 \dots \partial x_n}\,\mathrm{d}x = \int_{\mathbb{Q}^n} f(x)\,\mathrm{d}x - \sum_{j=1}^k a_j f(x^j) \tag{6.45}$$

for any

$$f \in S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)^{\urcorner}, \quad 0 < p,q \le \infty,\ \tfrac{1}{p} < r < 1 + \min\big(\tfrac{1}{p}, 1\big). \tag{6.46}$$

Proof. Step 1. If f is a smooth function with $f|\partial\mathbb{Q}_1^n = 0$ then (6.45) follows from integration by parts. If $f = v_{km}$ with (6.43) then it can be approximated by smooth functions vanishing at $\partial\mathbb{Q}_1^n$. Hence one has (6.45) for finite linear combinations of Faber functions from (6.43).

Step 2. By the comments in Remark 6.8 below it is sufficient to prove (6.45) for functions

$$f \in S^r_{pq}B(\mathbb{Q}^n)^{\urcorner} \quad \text{with } 1 < p,q < \infty,\ 1/p < r < 1, \tag{6.47}$$

where we used (6.26). The rest is a matter of embeddings as illustrated in Figure 4.4, p. 198. By (6.41) finite linear combinations of Faber functions v_{km} with (6.43) are dense in the spaces in (6.47). Next we use the isomorphic map (6.34), the duality (6.36) with $r-1$ in place of r and $\mathrm{disc}_{\Gamma,A} \in S^{1-r}_{p'q'}B(\mathbb{Q}^n)$, Figure 3.1, p. 127, which illustrates also the above situation. Then (6.45) for f with (6.47) follows by approximation from Step 1 and (6.44). □

Remark 6.8. We used in Step 2 that it is sufficient to deal with functions belonging to the spaces in (6.47). By (3.211) one has

$$S^r_{\infty q}\mathfrak{B}(\mathbb{Q}^n) \hookrightarrow S^r_{pq}B(\mathbb{Q}^n), \quad 1 < p < \infty,\ 0 < q \le \infty. \tag{6.48}$$

Otherwise one can rely on the well-known embedding theorems for the spaces $S^r_{pq}B(\mathbb{R}^n)$ according to [ST87, pp. 89/90, 131/132], extended from $\mathbb{R}^2$ to $\mathbb{R}^n$ and restricted afterwards to $\mathbb{Q}^n$. This shows also that (6.45) remains valid even for all functions

$$f \in S^r_{pq}A(\mathbb{Q}^n), \quad f|\partial\mathbb{Q}_1^n = 0 \quad \text{with } 0 < p,q \le \infty,\ r > 1/p, \tag{6.49}$$

where $S^r_{pq}A(\mathbb{Q}^n)$ are the spaces as introduced in Definition 1.56 ($p < \infty$ for F-spaces) and $\partial\mathbb{Q}_1^n$ has the same meaning as in (3.182). Here the case $S^r_{\infty\infty}B(\mathbb{Q}^n)$ is covered by (6.26), (6.48), and the corresponding assertion for $S^r_{\infty q}B(\mathbb{Q}^n)$, $0 < q < \infty$, follows by embedding. Of special interest will be (6.45) for

$$f \in S^1_pW(\mathbb{Q}^n)^{\urcorner}, \quad 1 < p < \infty. \tag{6.50}$$

By (3.196) we have in addition

$$S^1_{p,\min(p,2)}B(\mathbb{Q}^n)^{\urcorner} \hookrightarrow S^1_pW(\mathbb{Q}^n)^{\urcorner} \hookrightarrow S^1_{p,\max(p,2)}B(\mathbb{Q}^n)^{\urcorner} \hookrightarrow C(\mathbb{Q}^n). \tag{6.51}$$

Remark 6.9. The identity (6.45) for smooth functions and also for $f \in S^1_pW(\mathbb{Q}^n)^{\urcorner}$ with $1 < p < \infty$ is a special case of the famous Hlawka–Zaremba formula, [Hla61], [Zar68]. It plays a crucial role in the recent theory of numerical integration, discrepancy and tractability. We assumed $f|\partial\mathbb{Q}_1^n = 0$. This eliminates boundary values. But this is not necessary. If $f \in S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)$ or $f \in S^1_pW(\mathbb{Q}^n)$ belongs to the full spaces then (6.45) must be complemented by boundary terms. We refer to [NoW08, Section 3.1.5, pp. 34–37] and [NoW09, Section 9.8].

Remark 6.10. In Remark 6.2 and in Section 6.1.2 we discussed briefly the one-dimensional case. One may consider (5.105) as a one-dimensional version of (6.45). Otherwise we refer to Section 6.3.2 below where we discuss the one-dimensional counterpart of (6.45) in some detail.

6.2.3 Equivalences

Let again $\mathbb{Q}^n$ be the unit cube (6.1) in $\mathbb{R}^n$, $n \geq 2$. Let

$$1 \leq p, q \leq \infty, \qquad \frac{1}{p} - 1 < r < \frac{1}{p}, \tag{6.52}$$

Figure 2.3, p. 82. Then it follows from Proposition 6.3 that the discrepancy numbers $\mathrm{disc}_k \big(S^r_{pq} B(\mathbb{Q}^n) \big)$ in (6.7) are well defined. With

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1 \quad \text{one has} \quad 1 \leq p', q' \leq \infty, \qquad \frac{1}{p'} < 1 - r < 1 + \frac{1}{p'}. \tag{6.53}$$

In modification of (6.25), (6.44) one obtains that

$$S^{1-r}_{p'q'} \mathfrak{B}(\mathbb{Q}^n)^{\urcorner} \hookrightarrow \big\{ g \in C(\mathbb{Q}^n) : g | \partial \mathbb{Q}^n_1 = 0 \big\}. \tag{6.54}$$

Let $\mathrm{Int}_k \big(S^{1-r}_{p'q'} \mathfrak{B}(\mathbb{Q}^n)^{\urcorner} \big)$ be the corresponding integral numbers according to (5.5). Let $S^1_p W(\mathbb{Q}^n)^{\urcorner}$ with $1 < p < \infty$ be the Sobolev spaces according to (3.191).

Theorem 6.11. (i) *Let p, q, r be as in (6.52) with $q < \infty$ if $p = 1$ and $q > 1$ if $p = \infty$. Then*

$$\mathrm{disc}_k \big(S^r_{pq} B(\mathbb{Q}^n) \big) \sim \mathrm{Int}_k \big(S^{1-r}_{p'q'} \mathfrak{B}(\mathbb{Q}^n)^{\urcorner} \big), \quad k \in \mathbb{N}. \tag{6.55}$$

(ii) *Let $1 < p < \infty$. Then*

$$\mathrm{disc}_k \big(L_p(\mathbb{Q}^n) \big) \sim \mathrm{Int}_k \big(S^1_{p'} W(\mathbb{Q}^n)^{\urcorner} \big), \quad k \in \mathbb{N}. \tag{6.56}$$

Proof. *Step* 1. We prove part (i). One has by (6.53) and (6.33), (6.34) the isomorphic map

$$\frac{\partial^n}{\partial x_1 \ldots \partial x_n} S^{1-r}_{p'q'} \mathfrak{B}(\mathbb{Q}^n)^{\urcorner} = S^{-r}_{p'q'} \mathcal{B}(\mathbb{Q}^n). \tag{6.57}$$

By (6.35), (6.36) and (6.31) applied to $S^{-r}_{p'q'} B(\mathbb{Q}^n)$ we have

$$S^r_{pq} B(\mathbb{Q}^n)' = S^{-r}_{p'q'} B(\mathbb{Q}^n) = S^{-r}_{p'q'} \mathcal{B}(\mathbb{Q}^n), \quad 1 \leq p, q < \infty, \ \tfrac{1}{p} - 1 < r < \tfrac{1}{p}, \tag{6.58}$$

and

$$S^{-r}_{p'q'} \mathcal{B}(\mathbb{Q}^n)' = S^{-r}_{p'q'} B(\mathbb{Q}^n)' = S^r_{pq} B(\mathbb{Q}^n), \quad 1 < p, q \leq \infty, \ \tfrac{1}{p} - 1 < r < \tfrac{1}{p}. \tag{6.59}$$

This covers all cases in (6.52) with exception of $(p = 1, q = \infty)$ and $(p = \infty, q = 1)$. Taking on the right-hand side of (6.45) the supremum over the unit ball in $S^{1-r}_{p'q'} \mathfrak{B}(\mathbb{Q}^n)^{\urcorner}$ one obtains the integral numbers in (6.55). Then the isomorphic map (6.57) and the dual pairings (6.58), (6.59) applied to (6.45) prove (6.55).

Step 2. We prove part (ii). By Proposition 3.22 we have

$$\frac{\partial^n}{\partial x_1 \dots \partial x_n} S^1_{p'} W(\mathbb{Q}^n)^\urcorner = L_{p'}(\mathbb{Q}^n), \quad 1 < p < \infty, \; \tfrac{1}{p} + \tfrac{1}{p'} = 1. \tag{6.60}$$

Obviously, $L_p(\mathbb{Q}^n)' = L_{p'}(\mathbb{Q}^n)$ in the usual interpretation. Then one obtains (6.56) in the same way as in Step 1. □

Remark 6.12. Let A be a Banach space and let $c_0(A)$ be the subspace of $\ell_\infty(A)$ consisting of all sequences $\{a_j\}_{j=1}^\infty \subset A$ such that $\|a_j \,|A\| \to 0$ if $j \to \infty$. Then one has $c_0(A)' = \ell_1(A')$ for the dual space in the usual interpretation, [T78, Section 1.11.1, p. 68]. Based on this modification one can extend (6.37) to $p = \infty$ and/or $q = \infty$. This may pave the way to incorporate in part (i) of the above theorem the limiting cases $p = 1, q = \infty$ and $p = \infty, q = 1$, excluded so far.

6.3 Discrepancy numbers

6.3.1 Main assertions

Let again $\mathbb{Q}^n$, $n \geq 2$, be the unit cube in $\mathbb{R}^n$ according to (6.1). Let $\mathrm{disc}_k\big(S^r_{pq}A(\mathbb{Q}^n)\big)$ and $\mathrm{disc}_k\big(L_p(\mathbb{Q}^n)\big)$ be the discrepancy numbers introduced in Definition 6.1. In Proposition 6.3 we characterised the admitted spaces which cover, of course, the spaces in Theorem 6.11.

Theorem 6.13. *Let $\mathbb{Q}^n$ be the unit cube (6.1) in $\mathbb{R}^n$, $n \geq 2$.*

(i) *Let*

$$1 \leq p, q \leq \infty, \quad \frac{1}{p} - 1 < r < \frac{1}{p}, \tag{6.61}$$

Figure 2.3, p. 82 with $q < \infty$ if $p = 1$ and $q > 1$ if $p = \infty$. Then

$$c_1\, k^{r-1} (\log k)^{\frac{n-1}{q}} \leq \mathrm{disc}_k\big(S^r_{pq}B(\mathbb{Q}^n)\big) \leq c_2\, k^{r-1} (\log k)^{(n-1)(\frac{1}{q}+1-r)} \tag{6.62}$$

for some $c_1 > 0$, $c_2 > 0$ and all $k \in \mathbb{N}$, $k \geq 2$.

(ii) *Let $1 < p < \infty$. Then*

$$c_1\, k^{-1} (\log k)^{\frac{n-1}{2}} \leq \mathrm{disc}_k\big(L_p(\mathbb{Q}^n)\big) \leq c_2\, k^{-1} (\log k)^{3\frac{n-1}{2}} \tag{6.63}$$

for some $c_1 > 0$, $c_2 > 0$ and all $k \in \mathbb{N}$, $k \geq 2$.

Proof. Step 1. We prove part (i). The estimates from below in (5.90) rely on the extremal functions in (5.96) generalising (4.149), (4.153) from $n = 2$ to $n \geq 3$. Recall that $\partial \mathbb{Q}^n_1$ is the upper-right part of the boundary $\partial \mathbb{Q}^n$ of $\mathbb{Q}^n$ according to (3.182). By (6.43) we may assume that the above-mentioned extremal functions belong to the subspaces of the spaces covered by part (i) of Theorem 5.15 with vanishing boundary values at $\partial\mathbb{Q}_1^n$. This shows that one has for the integral numbers of $S^{1-r}_{p'q'}\mathfrak{B}(\mathbb{Q}^n)^\urcorner$ with (6.53) the same estimates as for $S^{1-r}_{p'q'}\mathfrak{B}(\mathbb{Q}^n)$, hence

$$c_1\, k^{r-1}(\log k)^{\frac{n-1}{q}} \le \mathrm{Int}_k\big(S^{1-r}_{p'q'}\mathfrak{B}(\mathbb{Q}^n)^\urcorner\big) \le c_2\, k^{r-1}(\log k)^{(n-1)(\frac{1}{q}+1-r)} \tag{6.64}$$

for some $c_1 > 0$, $c_2 > 0$ and all $k \in \mathbb{N}$, $k \ge 2$. Now (6.62) follows from (6.55) and (6.64).

Step 2. We prove part (ii). Recall that

$$S^0_{p,\min(p,2)}B(\mathbb{Q}^n) \hookrightarrow L_p(\mathbb{Q}^n) \hookrightarrow S^0_{p,\max(p,2)}B(\mathbb{Q}^n), \quad 1 < p < \infty. \tag{6.65}$$

This follows by restriction of a corresponding assertion in $\mathbb{R}^n$, [ST87, Proposition 2, pp. 88/89] (extended from $n = 2$ to $2 \le n \in \mathbb{N}$). Let $1 < p_1 \le 2 \le p_2 < \infty$. Then one obtains from (6.62) with $r = 0$, Hölder's inequality, and the monotonicity of the discrepancy numbers that

$$\begin{aligned} \mathrm{disc}_k\big(L_{p_1}(\mathbb{Q}^n)\big) &\le \mathrm{disc}_k\big(L_{p_2}(\mathbb{Q}^n)\big) \le c\, \mathrm{disc}_k\big(S^0_{p_2,2}B(\mathbb{Q}^n)\big) \\ &\le c'\, k^{-1}(\log k)^{3\frac{n-1}{2}}, \quad k \in \mathbb{N},\ k \ge 2, \end{aligned} \tag{6.66}$$

and

$$\begin{aligned} k^{-1}(\log k)^{\frac{n-1}{2}} &\le c\, \mathrm{disc}_k\big(S^0_{p_1,2}B(\mathbb{Q}^n)\big) \le c'\, \mathrm{disc}_k\big(L_{p_1}(\mathbb{Q}^n)\big) \\ &\le c'\, \mathrm{disc}_k\big(L_{p_2}(\mathbb{Q}^n)\big), \quad k \in \mathbb{N},\ k \ge 2. \end{aligned} \tag{6.67}$$

This proves (6.63). □

We reduced the proof of (6.63) to (6.62), the monotonicity of $L_p(\mathbb{Q}^n)$ in p and the elementary embedding (6.65). In the same way one can complement the above theorem by corresponding assertions for the spaces $S^r_{pq}F(\mathbb{Q}^n)$. This will be briefly discussed in Remark 6.28 below where we justify in particular the monotonicity of $S^r_{pq}F(\mathbb{Q}^n)$ in p. At this moment we restrict ourselves to the most interesting case, the Sobolev spaces

$$S^r_pH(\mathbb{Q}^n) = S^r_{p,2}F(\mathbb{Q}^n), \quad 1 < p < \infty,\ r \in \mathbb{R}, \tag{6.68}$$

with dominating mixed smoothness. We refer to Remark 1.39 (iii), extended to $\mathbb{R}^n$ and restricted afterwards to $\mathbb{Q}^n$. In particular,

$$S^r_{p,\min(p,2)}B(\mathbb{Q}^n) \hookrightarrow S^r_pH(\mathbb{Q}^n) \hookrightarrow S^r_{p,\max(p,2)}B(\mathbb{Q}^n) \tag{6.69}$$

is the counterpart of (6.65).

Corollary 6.14. *Let* $1 < p < \infty$ *and* $\frac{1}{p} - 1 < r < \frac{1}{p}$. *Then*

$$c_1\, k^{r-1}(\log k)^{\frac{n-1}{2}} \le \mathrm{disc}_k\big(S^r_pH(\mathbb{Q}^n)\big) \le c_2\, k^{r-1}(\log k)^{(n-1)(\frac{3}{2}-r)} \tag{6.70}$$

for some $c_1 > 0$, $c_2 > 0$ *and all* $k \in \mathbb{N}$, $k \ge 2$.

Proof. This follows in the same way as in Step 2 of the proof of Theorem 6.13 with (6.69) in place of (6.65) and the above indicated monotonicity of $S^r_p H(\mathbb{Q}^n)$ in p. □

Remark 6.15. Let $\mathrm{disc}^*_k\big(L_p(\mathbb{Q}^n)\big)$ and $\mathrm{disc}_k\big(L_p(\mathbb{Q}^n)\big)$, $1 \le p \le \infty$, be the discrepancy numbers as introduced in Definition 6.1. Let $n \ge 2$. Then

$$\mathrm{disc}^*_k\big(L_2(\mathbb{Q}^n)\big) \sim k^{-1}(\log k)^{\frac{n-1}{2}}, \quad 2 \le k \in \mathbb{N}. \tag{6.71}$$

The estimate from below goes back to [Roth54]. We discussed this crucial observation in Remark 6.2, (6.11), where one finds also references for alternative proofs. The corresponding estimate from above is due to [Fro80], [Roth80]. The extension of (6.71) from $L_2(\mathbb{Q}^n)$ to $L_p(\mathbb{Q}^n)$, $1 < p < \infty$, hence

$$\mathrm{disc}^*_k\big(L_p(\mathbb{Q}^n)\big) \sim k^{-1}(\log k)^{\frac{n-1}{2}}, \quad 1 < p < \infty,\ 2 \le k \in \mathbb{N}, \tag{6.72}$$

goes back to [Schw77] (estimate from below) and [Chen80] (estimate from above). The estimates from below for $\mathrm{disc}_k\big(L_p(\mathbb{Q}^n)\big)$, $1 < p < \infty$, in

$$\mathrm{disc}_k\big(L_p(\mathbb{Q}^n)\big) \sim k^{-1}(\log k)^{\frac{n-1}{2}}, \quad 1 < p < \infty,\ 2 \le k \in \mathbb{N}, \tag{6.73}$$

are due to [Chen85], [Chen87], $p = 2$, and to [Tem90], [Tem03, Theorem 3.5, p. 375], $1 < p < \infty$. The corresponding estimate from above follows from (6.72) and

$$\mathrm{disc}_k\big(L_p(\mathbb{Q}^n)\big) \le \mathrm{disc}^*_k\big(L_p(\mathbb{Q}^n)\big), \quad 1 \le p \le \infty. \tag{6.74}$$

Theorem 6.13 (ii) provides a new proof for the estimates from below in (6.73) in the context of Faber bases and spaces with dominating mixed smoothness. The first step in this direction may be found in [Tri09]. There is little hope that our method can be improved such that one obtains (6.73). It seems to be a sophisticated art to find irregularly distributed points in $\mathbb{Q}^n$ producing the exact upper bounds for $\mathrm{disc}^*_k\big(L_p(\mathbb{Q}^n)\big)$, $1 < p < \infty$. We refer to [Roth80], [Fro80], [Chen80] and the explicit (but rather involved) constructions in [ChS02], [Skr06]. The recent book [DiP10] deals in detail with problems of this type. Our method produces distributions of points which are closely related to the Smolyak algorithm and the so-called hyperbolic cross. In a somewhat different, but nevertheless related context it had been shown in [Pla00] that one cannot expect to obtain sharp results in this way.

Remark 6.16. We return to Remark 5.17 and the consequences of Theorem 6.11 for numerical integration. So far we have (6.55) for the subspace (6.54) of $S^{1-r}_{p'q'}\mathfrak{B}(\mathbb{Q}^n)$. Similarly in (6.56). It is desirable that one has the same equivalences for the full spaces. We replace $1-r$, p', q' in (6.53) by r, p, q and ask whether

$$\mathrm{Int}_k\big(S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)\big) \sim \mathrm{Int}_k\big(S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)^{\urcorner}\big), \quad k \in \mathbb{N}, \tag{6.75}$$

where

$$1 \le p, q \le \infty, \quad \frac{1}{p} < r < 1 + \frac{1}{p}, \tag{6.76}$$

and

$$\operatorname{Int}_k\big(S^1_p W(\mathbb{Q}^n)\big) \sim \operatorname{Int}_k\big(S^1_p W(\mathbb{Q}^n)^{\urcorner}\big), \quad k \in \mathbb{N}, \tag{6.77}$$

where $1 < p < \infty$. We justify (6.77) and (almost) (6.75), (6.76). According to Proposition 3.22, complemented by the n-dimensional version of Definition 3.24, the space $S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)$ can be decomposed into an interior part and a boundary part

$$\operatorname{tr}_{\partial\mathbb{Q}^n} S^r_{pq}\mathfrak{B}(\mathbb{Q}^n) = S^r_{pq}\mathfrak{B}(\partial\mathbb{Q}^n). \tag{6.78}$$

Similarly for $S^1_p W(\mathbb{Q}^n)$ based on (3.195). By the n-dimensional versions of (3.146), (3.150) one can identify these boundary spaces with complemented subspaces of $S^r_{pq}\mathfrak{B}(\mathbb{Q}^n)$ where one has to multiply boundary expansions in terms of Faber basis with the starting Faber functions $v_{km} \in \mathbb{N}^n_{-1} \setminus \mathbb{N}^n_0$ in (3.178). Hence the integral numbers for the full space can be reduced to the integral numbers for the subspace with vanishing boundary values and the integral numbers for $S^r_{pq}\mathfrak{B}(\partial\mathbb{Q}^n)$. Similarly for $S^1_p W(\mathbb{Q}^n)$. In case of $S^1_p W(\mathbb{Q}^n)$ we have in addition (6.73) and a corresponding assertion for integral numbers according to (6.56). Induction by dimension proves (6.77) and, hence,

$$\operatorname{Int}_k\big(S^1_p W(\mathbb{Q}^n)\big) \sim k^{-1}(\log k)^{\frac{n-1}{2}}, \quad 2 \le k \in \mathbb{N},\ 1 < p < \infty. \tag{6.79}$$

This coincides with (5.107).

Remark 6.17. For the spaces $L_p(\mathbb{Q}^n)$, $1 < p < \infty$, one has the equivalences (6.72), (6.73), $n \ge 2$. The behaviour of $\operatorname{disc}_k\big(L_1(\mathbb{Q}^n)\big)$, $\operatorname{disc}^*_k\big(L_1(\mathbb{Q}^n)\big)$ and also of $\operatorname{disc}_k\big(L_\infty(\mathbb{Q}^n)\big)$, $\operatorname{disc}^*_k\big(L_\infty(\mathbb{Q}^n)\big)$ seems to be rather tricky. There are no final answers if $n \ge 3$. We collect some known assertions. According to [Nie92, Chapter 3] one has for all $1 \le p \le \infty$,

$$\operatorname{disc}^*_k\big(L_p(\mathbb{Q}^n)\big) \le c\, k^{-1}(\log k)^{n-1}, \quad k \in \mathbb{N},\ k \ge 2. \tag{6.80}$$

In [Nie92] it is also mentioned that one widely beliefs that the right-hand side of (6.80) is the correct order for $p = \infty$, hence

$$\operatorname{disc}^*_k\big(L_\infty(\mathbb{Q}^n)\big) \sim k^{-1}(\log k)^{n-1}, \quad k \in \mathbb{N},\ k \ge 2. \tag{6.81}$$

This conjecture can also be found in [DrT97, p. 39]. It does not contradict the more recent conjecture in [BLV08, p. 2473] that for any n with $2 \le n \in \mathbb{N}$ there is a constant $c > 0$ such that

$$\operatorname{disc}^*_k\big(L_\infty(\mathbb{Q}^n)\big) \ge c\, k^{-1}(\log k)^{n/2}, \quad k \in \mathbb{N},\ k \ge 2. \tag{6.82}$$

For $n = 2$ both (6.81), (6.82) are known, [Schw72]. Furthermore for any $3 \le n \in \mathbb{N}$ there is a number $\eta = \eta(n) > 0$ such that for some $c > 0$,

$$\operatorname{disc}^*_k\big(L_\infty(\mathbb{Q}^n)\big) \ge c\, k^{-1}(\log k)^{\frac{n-1}{2}+\eta}, \quad k \in \mathbb{N},\ k \ge 2, \tag{6.83}$$

[BLV08, Theorem 2.4, p. 2473]. This shows at least that (6.72) cannot be extended to $p = \infty$. On the other hand it has been conjectured in [Lac08, p. 120] that (6.72) remains valid for $p = 1$, hence

$$\mathrm{disc}_k^*\big(L_1(\mathbb{Q}^n)\big) \sim k^{-1}(\log k)^{\frac{n-1}{2}}, \quad k \in \mathbb{N},\ k \geq 2. \tag{6.84}$$

This is known in case of $n = 2$, [Hal81]. But it seems to be unknown if $3 \leq n \in \mathbb{N}$.

6.3.2 The one-dimensional case, revisited

In Section 5.2 we dealt with integration on the unit interval $I = (0,1)$ in $\mathbb{R}$ with Theorem 5.7 as our main result. This is a more specific version of Theorem 5.4 applied to $\Omega = I$ and related comments in Remark 5.5. As far as discrepancies are concerned we inserted so far Proposition 6.5 to justify (6.10) in comparison with (6.11). It is quite clear that the above assertions about discrepancies for spaces in $\mathbb{Q}^n$, $2 \leq n \in \mathbb{N}$, have immediate and simpler counterparts for corresponding spaces on $\mathbb{Q}^1 = I = (0,1)$. But one can say a little bit more using some arguments which are not (or not yet) available in higher dimensions.

First we adapt Definition 6.1. Recall that $A^s_{pq}(I)$ are the spaces introduced in Definition 1.24 (i) as restrictions of $A^s_{pq}(\mathbb{R})$ to the unit interval $I = (0,1)$ in $\mathbb{R}$. Let χ_R be the characteristic function of the interval

$$R = \{x \in I : a < x < b\} \subset I, \tag{6.85}$$

where $0 \leq a < b \leq 1$. Let $\Gamma = \{x^j\}_{j=1}^k \subset I$ be a set of k points in I. Then χ_{R^j} is the characteristic function of the interval

$$R^j = \{x \in I : x^j < x < 1\}, \quad j = 1, \ldots, k. \tag{6.86}$$

Let $A = \{a_j\}_{j=1}^k \subset \mathbb{C}$. Then the *discrepancy functions*

$$\mathrm{disc}_{\Gamma,A}(x) = x - \sum_{j=1}^k a_j\, \chi_{R^j}(x), \quad x \in I, \tag{6.87}$$

and

$$\mathrm{disc}_{\Gamma}(x) = x - \frac{1}{k}\sum_{j=1}^k \chi_{R^j}(x), \quad x \in I, \tag{6.88}$$

are the one-dimensional versions of (6.4), (6.5). We modify Definition 6.1 as follows.

Definition 6.18. Let $G(I)$ be either $A^s_{pq}(I)$ with $0 < p,q \leq \infty$ ($p < \infty$ for F-spaces), $s \in \mathbb{R}$, such that

$$\chi_R \in A^s_{pq}(I) \quad \text{for any interval } R \tag{6.89}$$

according to (6.85) or $L_p(I)$, $1 \le p \le \infty$. Then

$$\mathrm{disc}_k\big(G(I)\big) = \inf \|\mathrm{disc}_{\Gamma,A}\,|G(I)\|, \quad k \in \mathbb{N}, \tag{6.90}$$

where the infimum is taken over all $\Gamma = \{x^j\}_{j=1}^k \subset I$, $A = \{a_j\}_{j=1}^k \subset \mathbb{C}$, and

$$\mathrm{disc}^*_k\big(G(I)\big) = \inf \|\mathrm{disc}_{\Gamma}\,|G(I)\|, \quad k \in \mathbb{N}, \tag{6.91}$$

where the infimum is taken over all $\Gamma = \{x^j\}_{j=1}^k \subset I$.

Remark 6.19. This is the modified one-dimensional version of Definition 6.1. There is the following one-dimensional counterpart of Proposition 6.3:

One has $\chi_R \in B^s_{pq}(I)$ for any R in (6.85) if, and only if,

$$\begin{cases} 0 < p \le \infty,\ 0 < q \le \infty, & s < 1/p, \\ 0 < p \le \infty,\ q = \infty, & s = 1/p, \end{cases} \tag{6.92}$$

and $\chi_R \in F^s_{pq}(I)$ if, and only if,

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s < 1/p. \tag{6.93}$$

First we extend the one-dimensional version of (6.45) to limiting cases. According to (3.8) and (3.9) one has final assertions for embeddings of $A^s_{pq}(I)$ into $C(I)$. In particular,

$$B^{1/p}_{pq}(I) \hookrightarrow C(I), \quad 0 < p < \infty,\ 0 < q \le 1. \tag{6.94}$$

Any

$$f \in B^{1/p}_{pq}(I) \quad \text{with } f(1) = 0,\ 0 < p < \infty,\ 0 < q \le 1, \tag{6.95}$$

can be approximated by smooth, say, C^∞, functions f_l with

$$f_l \to f \text{ in } B^{1/p}_{pq}(I), \quad f_l(1) \to f(1) = 0,\ l \to \infty. \tag{6.96}$$

Integration by parts with respect f_l and (6.96) result in

$$-\int_I \mathrm{disc}_{\Gamma,A}(x)\, f'(x)\,\mathrm{d}x = \int_I f(x)\,\mathrm{d}x - \sum_{j=1}^k a_j\, f(x^j). \tag{6.97}$$

This is the counterpart of (6.45) in a limiting situation. We add a comment about the convergence of the left-hand side. By embedding it is sufficient to deal with $1 \le p < \infty$, $q = 1$. First we remark that

$$f \mapsto f' \ \text{ maps } \big\{g \in B^{\frac{1}{p}}_{p,1}(I) : g(1) = 0\big\} \ \text{ onto } \ B^{\frac{1}{p}-1}_{p,1}(I). \tag{6.98}$$

Let $\frac{1}{p} + \frac{1}{p'} = 1$. Then one obtains by (6.92) and (1.76) that

$$\mathrm{disc}_{\Gamma,A} \in B^{\frac{1}{p'}}_{p',\infty}(\mathbb{R}) = B^{\frac{1}{p}-1}_{p,1}(\mathbb{R})', \quad 1 \le p < \infty. \tag{6.99}$$

Hence the left-hand side of (6.97) makes sense as a dual pairing. This shows in the same (but simpler) way as in the proof of Theorem 6.11 that

$$\mathrm{disc}_k\big(B^{1/p'}_{p',\infty}(I)\big) \sim \mathrm{Int}_k\big(B^{1/p}_{p,1}(I)\big), \quad 1 \le p < \infty,\ k \in \mathbb{N}, \tag{6.100}$$

where it does not matter whether one deals on the right-hand side with the full space $B^{1/p}_{p,1}(I)$ or the corresponding subspace in (6.98). We return later on to these limiting cases. Obviously, (6.97) makes also sense for all (non-limiting) spaces

$$\big\{f \in A^s_{pq}(I) : f(1) = 0\big\}, \quad 0 < p,q \le \infty,\ s > 1/p, \tag{6.101}$$

($p < \infty$ for F-spaces). First we deal with the counterpart of Theorem 6.13.

Theorem 6.20. (i) *Let*

$$1 \le p \le \infty, \quad \frac{1}{p} - 1 < s < \frac{1}{p}, \quad 0 < q \le \infty, \tag{6.102}$$

($p < \infty$ for F-spaces), Figure 2.3, p. 82. Then

$$\mathrm{disc}_k\big(A^s_{pq}(I)\big) \sim \mathrm{disc}^*_k\big(A^s_{pq}(I)\big) \sim k^{s-1}, \quad k \in \mathbb{N}. \tag{6.103}$$

(ii) *Let $1 \le p \le \infty$. Then there are two positive numbers c_1 and c_2 such that*

$$c_1\,k^{-1} \le \mathrm{disc}_k\big(L_1(I)\big) \le \mathrm{disc}_k\big(L_p(I)\big) \le \mathrm{disc}_k\big(L_\infty(I)\big) \le c_2\,k^{-1}, \tag{6.104}$$

$k \in \mathbb{N}$, and

$$c_1\,k^{-1} \le \mathrm{disc}^*_k\big(L_1(I)\big) \le \mathrm{disc}^*_k\big(L_p(I)\big) \le \mathrm{disc}^*_k\big(L_\infty(I)\big) \le c_2\,k^{-1}, \tag{6.105}$$

$k \in \mathbb{N}$.

Proof. Step 1. We prove part (ii) where we inserted (6.105)=(6.22) for sake of completeness. The right-hand side of (6.104) follows from (6.105). The monotonicity in p is a consequence of (6.90). As for the left-hand side of (6.104) we may assume that $k = 2^l$, $l \in \mathbb{N}$. Let $\Gamma = \{x^j\}_{j=1}^{2^l} \subset I$. Then $\Gamma \cap I^l_m = \emptyset$ for at least 2^l intervals $I^l_m = \big(m\,2^{-l-1}, (m+1)\,2^{-l-1}\big)$, $m = 0,\ldots,2^{l+1}-1$. The contribution of these intervals gives similarly as in (6.23) that

$$\int_0^1 \Big| x - \sum_{j=1}^{2^l} a_j\,\chi_{R^j}(x)\Big|\,\mathrm{d}x \ge c\,2^{-2l}2^l = c\,2^{-l} \tag{6.106}$$

for some $c > 0$ and all $l \in \mathbb{N}$. This proves the left-hand side of (6.104).

Step 2. We prove (6.103) with $A = B$ for the numbers disc_k and $q > 1$. Let $1 \le p,q \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. Then

$$B^s_{pq}(I)' = B^{-s}_{p'q'}(I), \quad 1 \le p,q < \infty,\ 1/p - 1 < s < 1/p, \tag{6.107}$$

and

$$B^{-s}_{p'q'}(I)' = B^s_{pq}(I), \quad 1 < p, q \le \infty, \quad 1/p - 1 < s < 1/p, \tag{6.108}$$

dual pairings in the context of $\big(D(I), D'(I)\big)$, are the counterparts of (6.58), (6.59), based on (6.36). This can be proved by reduction to sequence spaces according to Theorem 2.13 similarly (but simpler) as in (6.35)–(6.40). Let $\overset{\circ}{B}{}^s_{pq}(I)$ be the completion of $D(I)$ in $B^s_{pq}(I)$. Then

$$\overset{\circ}{B}{}^{-s}_{\infty,q'}(I)' = B^s_{1,q}(I), \quad 1 \le q \le \infty,\ 0 < s < 1. \tag{6.109}$$

The $\mathbb{R}$-counterpart may be found in [T83, (12), p. 180]. Reduction to I gives (6.109). But it can also be proved using $c_0(A)' = \ell_1(A')$, where A is a Banach space and c_0 is the subspace of ℓ_∞ consisting of all sequences converging to zero, [T78, Lemma, p. 68]. For this purpose one has to modify Theorem 2.13 appropriately. By (6.107), (6.108) one is in the same situation as in the proof of Theorem 6.11 with (6.97) in place of (6.45). Using in addition Theorem 5.7 one obtains that

$$\mathrm{disc}_k\big(B^s_{pq}(I)\big) \sim \mathrm{Int}_k\big(B^{1-s}_{p'q'}(I)\big) \sim k^{s-1}, \quad k \in \mathbb{N}, \tag{6.110}$$

for all spaces $B^s_{pq}(I)$ with

$$1 \le p, q \le \infty, \quad \frac{1}{p} - 1 < s < \frac{1}{p}, \tag{6.111}$$

$q < \infty$ if $p = 1$ and $q > 1$ if $p = \infty$. Using the above arguments and (6.109) then it follows that

$$\mathrm{disc}_k\big(B^s_{1,\infty}(I)\big) \sim \mathrm{Int}_k\big(B^{1-s}_{\infty,1}(I)^\circ\big) \sim k^{s-1}, \tag{6.112}$$

where $B^{1-s}_{\infty,1}(I)^\circ$ is the completion of smooth functions in $B^{1-s}_{\infty,1}(I)$. The last equivalence is a modification of Theorem 5.7 proved in the same way as there. One obtains (6.103) with $A = B$ for disc_k and $q > 1$.

Step 3. By Corollary 5.8 one has common best approximations for integral numbers. This is transferred by (6.97) to best approximating discrepancy functions. Then one can apply interpolation arguments. Recall that

$$B^s_{pq}(I) = \big(B^{s_0}_{pq_0}(I), B^{s_1}_{pq_1}(I)\big)_{\theta,q}, \quad 0 < \theta < 1, \tag{6.113}$$

$1 \le p \le \infty$, $1 < q_0 = q_1 \le \infty$, $0 < q \le 1$, $s_0 \ne s_1$, $s = (1-\theta)s_0 + \theta s_1$. Using Step 2 one obtains that

$$\mathrm{disc}_k\big(B^s_{pq}(I)\big) \le c\, k^{s-1}, \quad k \in \mathbb{N}, \tag{6.114}$$

for all p, q, s according to (6.102). The estimate from below for the remaining cases with $q \le 1$ is now a matter of monotonicity in q and of what we already know. This proves (6.103) for $A = B$ and disc_k. The corresponding assertion for $A = F$ follows from (4.88).

Step 4. According to Corollary 5.8 transferred via (6.97) to discrepancy functions the above arguments apply also to disc^*_k in place of disc_k. This gives finally part (i) of the theorem. □

Remark 6.21. Discrepancy in one dimension is surely less interesting than in higher dimensions in the context of spaces with dominating mixed smoothness. But it may serve as a model case. The restriction $p \geq 1$ in (6.102) comes from duality arguments based on (6.107), (6.108). According to Theorem 5.7 and Corollary 5.8 there is no such restriction for integral numbers. In any case if $0 < p \leq \infty$ then $s < \min\big(\frac{1}{p}, 1\big)$ is natural for discrepancy numbers. This follows from (6.92) if $p \geq 1$ and from (6.103) if $p < 1$. There remain the limiting cases $0 < s = \frac{1}{p} \leq 1$ which will be treated below. On the other hand the restriction $s > \frac{1}{p} - 1$ depends on our method, based on (6.110) and the corresponding restrictions of smoothness from above in Theorem 5.7. If one has (5.33) for larger values of s then one can modify (6.97) and extend (6.103) to $s \leq \frac{1}{p} - 1$ where $1 \leq p \leq \infty$. Equivalences of type (5.33) for integral numbers with larger values of s are known since a long time at least in some spaces as a consequence of diverse approximation methods. One may consult the references in Remark 5.5. The underlying techniques have little in common with our methods. But there may be a possibility to incorporate assertions of the desired type in the above context: One can try to replace the Faber expansions (5.28), (5.29) which cause in Theorem 5.7 the restriction $s < 1 + \frac{1}{p}$ by higher Faber splines according to Definition 3.38. Then one has Theorem 3.40 which applies to s restricted by (3.284). One may compare the Figures 3.1, p. 127, and 3.3, p. 170.

Finally we glance at the limiting cases in (6.92) and (6.94).

Proposition 6.22. *Let* $1 \leq p \leq \infty$. *Then*

$$\mathrm{disc}_k\big(B^{1/p}_{p,\infty}(I)\big) \sim \mathrm{disc}^*_k\big(B^{1/p}_{p,\infty}(I)\big) \sim k^{\frac{1}{p}-1}, \quad k \in \mathbb{N}, \tag{6.115}$$

and

$$\mathrm{Int}_k\big(B^{1/p}_{p,1}(I)\big) \sim k^{-1/p}. \tag{6.116}$$

Proof. *Step* 1. Let

$$f_k(x) = x - \frac{1}{k}\sum_{j=1}^{k} \chi_{R^j}(x), \quad x \in I, \tag{6.117}$$

be the same functions as at the beginning of the proof of Proposition 6.5 with

$$\mathrm{disc}^*_k\big(B^0_{\infty\infty}(I)\big) \leq \mathrm{disc}^*_k\big(L_\infty(I)\big) \leq c\, k^{-1}, \quad k \in \mathbb{N}, \tag{6.118}$$

for some $c > 0$. We used $L_\infty(I) \hookrightarrow B^0_{\infty\infty}(I)$, [T83, Proposition 2.5.7, p. 89] or

[ET96, p. 44]. Furthermore,

$$\begin{aligned} \| f_k \,|B^1_{1,\infty}(I)\| &\sim \| f_k \,|B^0_{1,\infty}(I)\| + \| f_k' \,|B^0_{1,\infty}(I)\| \\ &\le \Big\| x\, \chi_I(x) - \frac{1}{k}\sum_{j=1}^{k} \chi_{R^j}(x) \,|B^0_{1,\infty}(\mathbb{R})\Big\| \\ &\quad + \Big\| \chi_I - \frac{1}{k}\sum_{j=1}^{k} \delta_{x^j} \,|B^0_{1,\infty}(\mathbb{R})\Big\|, \end{aligned} \tag{6.119}$$

where χ_I is the characteristic function of I and δ_{x^j} is the δ-distribution with off-point x^j. If μ is a positive Radon measure with supp $\mu \subset \bar{I}$, then $\|\mu \,|B^0_{1,\infty}(\mathbb{R})\| \sim \mu(\mathbb{R})$ (total mass of μ). We refer to [T06, Proposition 1.127, p. 82] (and to [Kab08] for a generalisation). Applied to (6.119) one has

$$\| f_k \,|B^1_{1,\infty}(I)\| \le c, \quad k \in \mathbb{N}, \tag{6.120}$$

for some $c > 0$ which is independent of k. By (1.9) (extension from I to $\mathbb{R}$ and Hölder's inequality) one obtains that

$$\| f_k \,|B^{1/p}_{p,\infty}(I)\| \le c\, \| f_k \,|B^0_{\infty,\infty}(I)\|^{1-\frac{1}{p}} \| f_k \,|B^1_{1,\infty}(I)\|^{\frac{1}{p}} \le c'\, k^{\frac{1}{p}-1}, \tag{6.121}$$

where we used (6.118), based on (6.117) and (6.120). Hence

$$\operatorname{disc}_k\big(B^{1/p}_{p,\infty}(I)\big) \le \operatorname{disc}^*_k\big(B^{1/p}_{p,\infty}(I)\big) \le c\, k^{\frac{1}{p}-1}, \quad k \in \mathbb{N}, \tag{6.122}$$

$1 \le p \le \infty$. Recall that $B^s_{u,t}(I) \hookrightarrow B^s_{v,t}(I)$ for $s \in \mathbb{R}$ and u with $v \le u \le \infty$. Then it follows from the monotonicity of the integral numbers and Theorem 5.7 that for some $c > 0$

$$\operatorname{Int}_k\big(B^{1/p'}_{p',1}(I)\big) \ge c\, k^{\frac{-1}{p'}}, \quad \tfrac{1}{p} + \tfrac{1}{p'} = 1,\ 1 < p \le \infty. \tag{6.123}$$

Together with (6.100) one obtains that

$$\operatorname{disc}_k\big(B^{1/p}_{p,\infty}(I)\big) \ge c\, k^{\frac{1}{p}-1}, \quad k \in \mathbb{N},\ 1 < p \le \infty. \tag{6.124}$$

Together with (6.122) one obtains (6.115) for all p with $1 < p \le \infty$. Now (6.116) follows from (6.100) for all p with $1 \le p < \infty$.

Step 2. It remains to prove that for some $c > 0$,

$$\operatorname{disc}_k\big(B^1_{1,\infty}(I)\big) \ge c, \quad k \in \mathbb{N}, \tag{6.125}$$

and

$$\operatorname{Int}_k\big(B^0_{\infty,1}(I)\big) \ge c, \quad k \in \mathbb{N}. \tag{6.126}$$

Let $\Gamma = \{x^j\}_{j=1}^k \subset I$ and let as before χ_{R^j} be the characteristic functions of the intervals $(x^j, 1)$. Let

$$f_k(x) = x - \sum_{j=1}^{k} a_j \, \chi_{R^j}(x), \quad x \in I, \; \{a_j\}_{j=1}^k \subset \mathbb{C}. \tag{6.127}$$

One has similarly as in (6.119) that

$$\| f_k \,| B^1_{1,\infty}(I)\| \geq c \, \Big\| \chi_I - \sum_{j=1}^{k} a_j \, \delta_{x^j} \,| B^0_{1,\infty}(\mathbb{R}) \Big\| \tag{6.128}$$

for some $c > 0$ which is independent of k. In Remark 6.23 below we justify this estimate and that one can work with spaces on $\mathbb{R}$ instead of I. Let $\varphi \in D(\mathbb{R})$, $\int_{\mathbb{R}} \varphi(y)\,\mathrm{d}y = 0$, $\varphi(0) = 1$. By Theorem 1.7,

$$g_{\Gamma,l}(x) = \sum_{j=1}^{k} \varepsilon_j \, \varphi\big(2^l(x - x^j)\big), \quad l \in \mathbb{N}, \; \varepsilon_j \in \mathbb{C}, \; |\varepsilon_j| = 1, \tag{6.129}$$

is an atomic decomposition in $B^0_{\infty,1}(\mathbb{R})$ with

$$\|g_{\Gamma,l} \,| B^0_{\infty,1}(\mathbb{R})\| \leq c, \quad l \in \mathbb{N}. \tag{6.130}$$

For given Γ we choose $l \in \mathbb{N}$ large. Let $\varepsilon_j a_j = |a_j|$. Then it follows by duality (or Hölder's inequality) that

$$\sum_{j=1}^{k} |a_j| \sim \Big(\chi_I - \sum_{j=1}^{k} a_j \, \delta_{x^j} \,, g_{\Gamma,l} \Big) \leq c \, \Big\| \chi_I - \sum_{j=1}^{k} a_j \, \delta_{x^j} \,| B^0_{1,\infty}(\mathbb{R}) \Big\|. \tag{6.131}$$

On the other hand,

$$\Big\| \chi_I - \sum_{j=1}^{k} a_j \, \delta_{x^j} \,| B^0_{1,\infty}(\mathbb{R}) \Big\| \geq c' - c' \sum_{j=1}^{k} |a_j| \tag{6.132}$$

for some $c' > 0$. Hence the right-hand side of (6.128) is uniformly bounded from below by a positive constant. This proves (6.125). Finally we apply (6.97) to the dual pairing of $B^1_{1,\infty}(\mathbb{R})$ and the completion of $D(\mathbb{R})$ in $B^{-1}_{\infty,1}(\mathbb{R})$. We refer again to [T83, (12), p. 180]. Then (6.126) follows from (6.125). □

Remark 6.23. Let $\psi \in D(I)$. Then one obtains from the pointwise multiplier property that

$$\Big\| \psi(x) \Big(\chi_I - \sum_{j=1}^{k} a_j \, \delta_{x^j} \Big) | B^0_{1,\infty}(\mathbb{R}) \Big\| \leq c \, \Big\| \chi_I - \sum_{j=1}^{k} a_j \, \delta_{x^j} | B^0_{1,\infty}(I) \Big\|. \tag{6.133}$$

This is sufficient to justify the estimate from below as in (6.128).

Remark 6.24. The limiting case (6.116) with $p = 1$ follows also from

$$B^1_{p,1}(I) \hookrightarrow B^1_{1,1}(I) \hookrightarrow W^1_1(I), \quad 1 < p < \infty, \tag{6.134}$$

(5.33) and (5.104).

6.3.3 Comments, problems, proposals

We add a few remarks. Some of them may be considered as proposals for future research.

Remark 6.25. The literature mentioned in Remarks 6.15, 6.17 deals almost exclusively with discrepancies in $L_p(\mathbb{Q}^n)$, $1 \le p \le \infty$. The only exception known to us is [Lac08] where it is shown that

$$\mathrm{disc}^*_k\big(L_1(\log L)_{\frac{n-2}{2}}(\mathbb{Q}^n)\big) \sim k^{-1}(\log k)^{\frac{n-1}{2}}, \quad 2 \le k \in \mathbb{N}, \tag{6.135}$$

$2 \le n \in \mathbb{N}$. Here $L_1(\log L)_{\frac{n-2}{2}}(\mathbb{Q}^n)$ are the well-known Zygmund spaces. They are smaller than $L_1(\mathbb{Q}^n)$ but larger than any $L_p(\mathbb{Q}^n)$ with $p > 1$. Details about these spaces may be found in [ET96, Section 2.6.1, pp. 65–69]. The equivalence (6.135) supports the conjecture (6.84). In the present book we dealt with numerical integration and discrepancy based on Haar bases, Faber bases and sampling. A first description of this (as we hope) new method may be found in [Tri09]. Although we discussed in this paper discrepancies preferably in $L_p(\mathbb{Q}^n)$ we also indicated how this approach can be applied to other spaces, for example, $S^r_{pq}B(\mathbb{Q}^n)$. It is one of the main aims of this book to present this theory in detail. Both in Theorem 5.15 for integral numbers and in Theorem 6.13 for discrepancy numbers one has gaps in the log-powers between the estimates for below and from above. This is caused by our method. On the other hand, according to (6.73) the lower bounds are the exact bounds in case of the spaces $L_p(\mathbb{Q}^n)$, $1 < p < \infty$. One may ask whether this is also the case for the spaces covered by Theorem 6.13 (i). In other words,

if

$$1 \le p, q \le \infty, \quad \frac{1}{p} - 1 < r < \frac{1}{p}, \tag{6.136}$$

then

$$\mathrm{disc}_k\big(S^r_{pq}B(\mathbb{Q}^n)\big) \sim k^{r-1}(\log k)^{\frac{n-1}{q}}, \quad 2 \le k \in \mathbb{N}, \tag{6.137}$$

is the natural complement of (6.73).

This is the counterpart of the question (5.108). From (5.109), or the right-hand side of (6.65) and (6.73), follows that

$$\mathrm{disc}_k\big(S^0_{p,2}B(\mathbb{Q}^n)\big) \le c\, k^{-1}(\log k)^{\frac{n-1}{2}}, \quad 1 < p \le 2,\ 2 \le k \in \mathbb{N},$$

which confirms (6.137) in this special case. As mentioned in Remark 6.15 the construction of optimally distributed points in $\mathbb{Q}^n$ resulting in the exact upper bounds in (6.73) is a rather sophisticated task. One may ask whether these distinguished sets of points can also be used to obtain the exact upper bounds in (6.137). From the structural point of view the spaces $S^r_{pq}B(\mathbb{Q}^n)$ are much simpler than $L_p(\mathbb{Q}^n)$. They are isomorphic to the handsome sequence spaces $s^H_{pq}b(\mathbb{Q}^n)$ in (2.317) and admit corresponding characterisations in terms of Haar tensor bases. By Theorem 6.11 and (6.75) there is a symbiotic relationship between discrepancy numbers and integral numbers at least if $1 \le p, q \le \infty$. In any case one can ask as in (5.108) whether the lower bounds in Theorem 5.15 are the exact bounds. Recently A. Hinrichs proved in [Hin09] that

$$\operatorname{disc}^*_k\big(S^r_{pq}B(\mathbb{Q}^2)\big) \sim \operatorname{disc}_k\big(S^r_{pq}B(\mathbb{Q}^2)\big) \sim k^{r-1}(\log k)^{1/q}, \quad 2 \le k \in \mathbb{N},$$

where $1 \le p, q \le \infty$, ($q < \infty$ if $p = 1$) and $0 \le r < 1/p$. Here $\operatorname{disc}^*_k\big(S^r_{pq}B(\mathbb{Q}^2)\big)$ is the obvious counterpart of (6.9). This confirms (6.137) in two dimensions under the indicated restrictions. We refer also to Remark 6.28 below. Using Theorem 6.11 and (6.75) one can transfer this assertion to some integral numbers in the square,

$$\operatorname{Int}_k\big(S^r_{pq}B(\mathbb{Q}^2)\big) \sim k^{-r}(\log k)^{1-\frac{1}{q}}, \quad 2 \le k \in \mathbb{N},$$

where $1 < p < \infty$, $1 \le q < \infty$, $1/p < r \le 1$ and

$$\operatorname{Int}_k S^r\mathcal{C}(\mathbb{Q}^2) \sim k^{-r}\ \log k, \quad 2 \le k \in \mathbb{N},\ 0 < r < 1.$$

Here $S^r\mathcal{C}(\mathbb{Q}^2) = S^r_{\infty\infty}B(\mathbb{Q}^2)$ are the Hölder spaces with dominating mixed first differences which can be intrinsically normed by (4.218). This confirms (5.108) partly.

Remark 6.26. We return to the one-dimensional case and the discussion in Remark 6.21. Theorem 5.7 is based on the Faber expansion (5.28), (5.29). This restricts the smoothness s by $s < 1 + \frac{1}{p}$. One can replace the Faber system $\{v_{jm}\}$ in (5.29) by the Faber spline systems $\{v^l_{jm}\}$ according to Definition 3.38. Then one has the expansion (3.285) in Theorem 3.40. It restricts s by (3.284), Figures 3.3, p. 170, in comparison with Figure 3.1, p. 127. This may serve as a basis to extend Theorem 5.7 to larger values of s in the context of our approach. Let $\widetilde{B}^s_{pq}(I)$ be the spaces introduced in Definition 1.24 (ii). Let

$$1 \le p, q < \infty, \quad \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1, \quad s \in \mathbb{R}. \tag{6.138}$$

Then $D(I)$ is dense in $\widetilde{B}^s_{pq}(I)$ and

$$\widetilde{B}^s_{pq}(I)' = B^{-s}_{p'q'}(I) \tag{6.139}$$

are dual spaces interpreted within the dual pairing $\big(D(I), D'(I)\big)$. Although this has been essentially known for a long time, [T78, Section 4.8.1, p. 332], we refer for a careful discussion to [T08, Section 3.3.4, pp. 97–99] (thirty years later). The step from integral numbers to discrepancy numbers relies on duality arguments based on

(6.97). Now one has to replace (6.107) by (6.139). For this purpose one can ask for a counterpart of (6.101) such that $f \mapsto f'$ maps this space onto $\widetilde{B}^s_{pq}(I)$ (or a substitute). This might be a way to extend Theorem 6.20 to $s \le \frac{1}{p} - 1$.

Remark 6.27. In Section 3.5.3, Points 3 and 4, we suggested to extend the above (one-dimensional) Faber spline systems now denoted by $\{v^L_{jm}\}$ to corresponding Faber tensor spline systems in higher dimensions. Then it might be possible to extend Theorem 5.15 from p, q, r as in (5.89) to

$$0 < p, q \le \infty, \quad \frac{1}{p} < r < 2L + 1 + \min\Big(\frac{1}{p}, 1\Big), \quad L \in \mathbb{N}_0. \tag{6.140}$$

The estimates for discrepancy numbers in Theorem 6.13 relied afterwards on (6.55) based on the identity (6.45) and the duality (6.36). It might well be possible to generalise these ingredients. Let

$$\widetilde{S}^r_{pq}B(\mathbb{Q}^n) = \big\{ f \in S^r_{pq}B(\mathbb{R}^n) : \operatorname{supp} f \subset \overline{\mathbb{Q}^n} \big\} \tag{6.141}$$

in generalisation of (2.299). Using the duality (2.272) (extended to $\mathbb{R}^n$), then

$$\widetilde{S}^r_{pq}B(\mathbb{Q}^n)' = S^{-r}_{p'q'}B(\mathbb{Q}^n) \tag{6.142}$$

with

$$r \in \mathbb{R}, \quad 1 \le p, q < \infty, \quad \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1, \tag{6.143}$$

is a reasonable extension of (6.35), (6.36). We refer again to [T78, Section 4.8.1, p. 332], [T08, Section 3.3.4] for counterparts of (6.142) in terms of the isotropic spaces $B^s_{pq}(\Omega)$. The adaption of (6.45) requires spaces which are mapped by $f \mapsto \frac{\partial^n f}{\partial x_1 \ldots \partial x_n}$ onto $\widetilde{S}^r_{pq}B(\mathbb{Q}^n)$. If all this works then one has a good chance to extend Theorem 6.13 (i) from p, q, r in (6.61) to

$$1 \le p, q \le \infty, \quad \frac{1}{p} - 1 - 2L < r < \frac{1}{p}, \quad L \in \mathbb{N}_0. \tag{6.144}$$

But nothing has been done so far and the above comments may be considered as suggestions for further research.

Remark 6.28. In Chapters 2, 3 we developed the theory of Haar bases and Faber bases both for B-spaces such as B^s_{pq}, $S^r_{pq}B$, and F-spaces such as F^s_{pq}, $S^r_{pq}F$, with the Sobolev spaces H^s_p, S^r_pH, including the classical Sobolev spaces W^k_p, S^k_pW, as special cases. Later on we gave preference to B-spaces, especially in Chapters 5, 6 about numerical integration and discrepancy. The corresponding assertions for the special F-spaces S^1_pW and L_p are included afterwards by elementary embeddings of type (4.158), (4.159) and its n-dimensional version (3.196). It is remarkable that these apparently rather crude elementary embeddings produce exact lower bounds for the integral numbers $\operatorname{Int}_k\big(S^1_pW(\mathbb{Q}^n)\big)$ and the discrepancy numbers $\operatorname{disc}_k\big(L_p(\mathbb{Q}^n)\big)$,

$1 < p < \infty$. It makes sense to expect that this surprising phenomenon is not restricted to the special F-spaces

$$L_p(\mathbb{Q}^n) = S^0_p H(\mathbb{Q}^n) = S^0_{p,2} F(\mathbb{Q}^n), \quad 1 < p < \infty, \tag{6.145}$$

and

$$S^1_p W(\mathbb{Q}^n) = S^1_p H(\mathbb{Q}^n) = S^1_{p,2} F(\mathbb{Q}^n), \quad 1 < p < \infty, \tag{6.146}$$

but applies also to larger classes of F-spaces. Recall that

$$S^r_{p,\min(p,q)} B(\mathbb{Q}^n) \hookrightarrow S^r_{pq} F(\mathbb{Q}^n) \hookrightarrow S^r_{p,\max(p,q)} B(\mathbb{Q}^n), \tag{6.147}$$

where $r \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$, [ST87, p. 89], [ScS04, p. 121]. The spaces $S^r_{pq} A(\mathbb{Q}^n)$ are defined by restriction of functions $g \in S^r_{pq} A(\mathbb{R}^n)$, where one may assume that all admitted g have compact support in a bounded domain Ω with $\overline{\mathbb{Q}^n} \subset \Omega$. Then the characterisation of g in terms of wavelets according to Theorem 1.54 and the monotonicity of $L_p(\mathbb{Q}^n)$ and $L_p(\Omega)$ with respect to p show that

$$S^r_{p_1 q} F(\mathbb{Q}^n) \hookrightarrow S^r_{qq} F(\mathbb{Q}^n) = S^r_{qq} B(\mathbb{Q}^n) \hookrightarrow S^r_{p_2 q} F(\mathbb{Q}^n), \tag{6.148}$$

$r \in \mathbb{R}$, $0 < p_2 \le q \le p_1 < \infty$. The elementary embeddings (6.147), (6.148) are the counterparts of (4.158), (4.159) where the role of 2 is taken over now by q. The bounds for sampling numbers, integral numbers and discrepancy numbers for $S^r_{pq} B(\mathbb{Q}^n)$ produce corresponding assertions for $S^r_{pq} F(\mathbb{Q}^n)$ and in particular for the Sobolev spaces $S^r_p H(\mathbb{Q}^n) = S^r_{p,2} F(\mathbb{Q}^n)$, $1 < p < \infty$. This can be done in the same way as in Theorems 5.15 (integral numbers) and 6.13 (discrepancy numbers). We restrict ourselves to $1 \le p, q < \infty$ ensuring in particular the n-dimensional version of (4.128) such that (6.147) and (6.148) can be applied. Then it follows by the same arguments as in the indicated theorems that for some $c_1 > 0$, $c_2 > 0$ and all $2 \le k \in \mathbb{N}$,

$$c_1\, k^{-r} (\log k)^{(n-1)(1-\frac{1}{q})} \le \mathrm{Int}_k \big(S^r_{pq} F(\mathbb{Q}^n) \big) \le c_2\, k^{-r} (\log k)^{(n-1)(r+1-\frac{1}{q})} \tag{6.149}$$

where

$$\frac{1}{p} < r < 1 + \frac{1}{p},$$

and

$$c_1\, k^{r-1} (\log k)^{\frac{n-1}{q}} \le \mathrm{disc}_k \big(S^r_{pq} F(\mathbb{Q}^n) \big) \le c_2\, k^{r-1} (\log k)^{(n-1)(\frac{1}{q}+1-r)} \tag{6.150}$$

where

$$\frac{1}{p} - 1 < r < \frac{1}{p}.$$

These are the same estimates as in Theorems 5.15 and 6.13 for $S^r_{pq} B(\mathbb{Q}^n)$. In case of the Sobolev spaces $S^r_p H(\mathbb{Q}^n) = S^r_{p,2} F(\mathbb{Q}^n)$ one has in particular

$$c_1\, k^{-r} (\log k)^{\frac{n-1}{2}} \le \mathrm{Int}_k \big(S^r_p H(\mathbb{Q}^n) \big) \le c_2\, k^{-r} (\log k)^{(n-1)(r+\frac{1}{2})} \tag{6.151}$$

where

$$1 < p < \infty, \quad \frac{1}{p} < r < 1 + \frac{1}{p},$$

and (also covered by Corollary 6.14)

$$c_1\, k^{r-1} (\log k)^{\frac{n-1}{2}} \le \mathrm{disc}_k\big(S^r_p H(\mathbb{Q}^n)\big) \le c_2 k^{r-1} (\log k)^{(n-1)(\frac{3}{2}-r)} \tag{6.152}$$

where

$$1 < p < \infty, \quad \frac{1}{p} - 1 < r < \frac{1}{p}.$$

This generalises also (5.91) (where $r = 1$, $q = 2$) and (6.63) (where $r = 0$, $q = 2$). In modification of (5.108) and (6.137) it makes sense to ask whether the lower bounds in (6.149), (6.150) are sharp. Using the recent assertions in [Hin09] briefly mentioned at the end of Remark 6.25 and the elementary embeddings (6.147), (6.148) then one can answer this question for some cases in two dimensions,

$$\mathrm{disc}_k\big(S^r_{pq} B(\mathbb{Q}^2)\big) \sim \mathrm{disc}_k\big(S^r_{pq} F(\mathbb{Q}^2)\big) \sim k^{r-1} (\log k)^{1/q}, \quad 2 \le k \in \mathbb{N},$$

where $1 < p, q < \infty$ and $0 \le r < 1/p$. This equivalence applies in particular to the Sobolev spaces,

$$\mathrm{disc}_k\big(S^r_p H(\mathbb{Q}^2)\big) \sim k^{r-1}\, (\log k)^{1/2}, \quad 1 < p < \infty,\ 0 \le r < 1/p,\ 2 \le k \in \mathbb{N}.$$

For the integral numbers one has

$$\mathrm{Int}_k\big(S^r_{pq} B(\mathbb{Q}^2)\big) \sim \mathrm{Int}_k\big(S^r_{pq} F(\mathbb{Q}^2)\big) \sim k^{-r}\, (\log k)^{1-\frac{1}{q}}, \quad 2 \le k \in \mathbb{N},$$

where $1 < p, q < \infty$ and $\frac{1}{p} < r \le 1$. With $q = 2$ one obtains for the Sobolev spaces

$$\mathrm{Int}_k\big(S^r_p H(\mathbb{Q}^2)\big) \sim k^{-r}\, (\log k)^{1/2}, \quad 1 < p < \infty,\ 1/p < r \le 1,\ 2 \le k \in \mathbb{N}.$$

The extension of this theory as indicated in Remarks 6.26, 6.27 to larger values of r for integral numbers and smaller values of r for discrepancy numbers can be carried over from B-spaces to F-spaces using again the above elementary embeddings.

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